Numerical Analysis of Ordinary Differential Equations Exercises

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Exercise sheet 12
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Exercise 12.1 (**Stability of BVP**)

On the Exercise Sheet 11 we considered for $k \in \mathbb{R}, k \neq 0$ the differential equation

$$u'' = k^2 u,$$

with the two linearly independent solutions $e^{k(t-t_0)}$ and $e^{-k(t-t_0)}$.
For $[a, b] = [-1, 1]$ and $u(a) = u(b) = 1$ we solved the corresponding boundary value problem.
To this end, we reformulated the BVP in first order form

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ k^2 u \end{pmatrix}, \tag{12.1}$$

$$B_a \begin{pmatrix} u(a) \\ v(a) \end{pmatrix} + B_b \begin{pmatrix} u(b) \\ v(b) \end{pmatrix} = \begin{pmatrix} g_a \\ g_b \end{pmatrix} \tag{12.2}$$

where $B_a = [1\ 0; 0\ 0]$, $B_b = [0\ 0; 1\ 0]$ and $g_a = g_b = 1$ and determined the fundamental matrix

$$W(t; t_0) = \begin{pmatrix} \cosh(k(t-t_0)) & \frac{1}{k}\sinh(k(t-t_0)) \\ k\sinh(k(t-t_0)) & \cosh(k(t-t_0)) \end{pmatrix}.$$

a) Use Theorem 5.3.10 to determine the maximal sensitivity on the interval $[a, b] = [-1, 1]$ of the solution $u(t)$ with respect to small perturbations of the boundary values g_a and g_b.

b) Use Theorem 5.3.11 to determine the maximal sensitivity on the interval $[a, b] = [-1, 1]$ of the solution $u(t)$ with respect to perturbations of the right hand side of the form $g(t, u(t))$. Make suitable assumptions on g.

c) Compute the sensitivities of the initial value problem given by (12.1) and the initial condition

$$\begin{pmatrix} u \\ v \end{pmatrix}(a) = \begin{pmatrix} 1 \\ \frac{k}{\sinh(2k)}(1-\cosh(2k)) \end{pmatrix} \tag{12.3}$$

(initial condition has been determined in Exercise 11.2) with respect to perturbations of the initial values and the right hand side according to Theorems 5.2.6 and 5.2.8.

d) Compare the results for initial and boundary value problems and their behavior for $k \to \infty$.

Exercise 12.2 (**Nonlinear solution in BDF methods**)
Consider an initial value problem with nonlinear ODE system and solve it with a higher order BDF methods

$$\sum_{r=0}^{s} \alpha_{s-r} y_{k-r} = h\beta_s f(t_k, y_k).$$

a) In each time step a nonlinear system of equations has to be solved. Formulate Newton's method for this system.

b) Newton's method needs an initial value. Which values are good guesses?

c) Write down an algorihtmic procedure for the efficient and error-controlled computation of the new approximation y_k. Reuse matrices in Newton's method as long as possible. What should be done first if the computation of y_k fails: An update of the matrix or a Reduction of the step size?