

## Homework No. 1 Numerical Methods for PDE, Winter 2016/17

### Problem 1.1: Variational equations in $\mathbb{R}^n$

Given a symmetric, positive definite matrix  $A \in \mathbb{R}^{n \times n}$  and a vector  $b \in \mathbb{R}^n$  and the “energy functional”

$$E(x) = \frac{1}{2}x^T Ax - x^T b, \quad (1.1)$$

- (a) Derive the variational equation of the minimization problem by studying the derivative of the auxiliary function  $\Phi(t) = E(x + ty)$  for arbitrary  $y \in \mathbb{R}^n$ .
- (b) Show that a vector  $x \in \mathbb{R}^n$  minimizes  $E(x)$ , that is,

$$E(x) \leq E(y) \quad \forall y \in \mathbb{R}^n,$$

if and only if

$$Ax = b.$$

- (c) Conclude that the minimizer  $x$  exists and is unique.

### Problem 1.2: Minimizing sequence

- (a) Show that a sequence  $\{x^{(k)}\}$  such that for the energy functional in (1.1) holds

$$E(x^{(k)}) \rightarrow \inf_{y \in \mathbb{R}^d} E(y), \quad (1.2)$$

necessarily converges to the minimizer  $x$  from Problem 1.1. The “binomial formula”  $x^T Ax - y^T Ay = (x + y)^T A(x - y)$  and the fact that  $A$  is invertible are useful ingredients to this proof.

- (b) Show without assuming the existence of the minimizer  $x$ , that a sequence  $\{x^{(k)}\}$ , for which (1.2) holds is necessarily a Cauchy sequence. Can you conclude the existence of a minimizer  $x$ ?

### Problem 1.3: Integration by parts

Let  $\Omega$  be a domain in  $\mathbb{R}^d$ . Use the Gauß theorem for smooth vector fields  $\varphi : \Omega \rightarrow \mathbb{R}^d$ , namely,

$$\int_{\Omega} \nabla \cdot \varphi \, dx = \int_{\partial\Omega} \varphi \cdot \mathbf{n} \, ds,$$

to show Green’s first and second formula (for smooth scalar functions  $u$  and  $v$ )

$$\begin{aligned} - \int_{\Omega} \Delta uv \, dx &= \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \partial_n uv \, ds \\ \int_{\Omega} (u\Delta v - v\Delta u) \, dx &= \int_{\partial\Omega} (u\partial_n v - v\partial_n u) \, ds. \end{aligned}$$

Here,  $\mathbf{n}$  is the outward unit normal vector to  $\Omega$  on  $\partial\Omega$ . The differential operators have the meaning:

$\nabla u = (\partial_1 u, \dots, \partial_d u)^T$	gradient
$\partial_n u = \mathbf{n} \cdot \nabla u$	normal derivative
$\nabla \cdot \varphi = \partial_1 \varphi_1 + \dots + \partial_d \varphi_d$	divergence
$\Delta u = \nabla \cdot \nabla u = \partial_{11} u + \dots + \partial_{dd} u$	Laplacian