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Homework No. 3
Numerical Methods for PDE, Winter 2016/17

Problem 3.1: Trilinearform

For $\Omega \subset \mathbb{R}^2$ consider the term

$$c(w; u, v) = (w \cdot \nabla u, v)\,, \quad c(\cdot; \cdot, \cdot) : W \times V \times V \to \mathbb{R}, \qquad \text{with } W = H_0^1(\Omega; \mathbb{R}^2), \quad \text{and} \quad V = H_0^1(\Omega).$$

Note: The semicolon indicates that we will later on use w as data, such that $c(w; ., .)$ is a bilinearform which fits into our existing framework.

(a) Show that $c(\cdot; \cdot, \cdot)$ is linear in each variable (altogether trilinear) and continuous.

Hint: Show that $|c(w; u, v)|$ is bounded. Use the Sobolev embedding theorem.

(b) Show that the following identity holds:

$$(w \cdot \nabla u, u) = -\frac{1}{2} \left(\nabla \cdot w, u^2\right).$$

Hint: The notation

$$\nabla \cdot \varphi := \partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2$$

denotes the divergence of a sufficiently regular vector field $\varphi(x) = \big(\varphi_1(x), \varphi_2(x)\big)^T$.

(c) Deduce an analogous formula for $w \in H^1(\Omega; \mathbb{R}^2)$ and $u \in H^1(\Omega)$ without zero boundary conditions.

Problem 3.2: Stationary convection-diffusion equation

Consider the stationary convection-diffusion equation

$$\begin{aligned} -\Delta u + w \cdot \nabla u &= f, && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega \end{aligned}$$

for a bounded domain $\Omega \subset \mathbb{R}^2$ and a given function $w \colon \Omega \to \mathbb{R}^2$ with $\nabla \cdot w = 0$.

(a) Formulate the problem weakly for functions $u \in H_0^1(\Omega)$.

(b) Show, that there is a unique solution of the weak formulation by using the theorem of Lax-Milgram. Make suitable assumptions on the data w and f. Why don't we use Riesz' representation theorem directly to show existence?

(c) What happens, if we do not assume $\nabla \cdot w = 0$? Can we still guarantee existence?