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Homework No. 4
Numerical Methods for PDE, Winter 2016/17

Problem 4.1: Elastic plates

"Plates" are flat bodies with (constant) positive, but small thickness. In linear Kirchhoff plate theory the vertical displacement of a plate can be described as solution u of the following fourth order pde

$$\Delta^2 u = f, \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain and Δ^2 is the **biharmonic operator** defined by

$$\Delta^2 u(x,y) = \Delta(\Delta u(x,y)) = \partial_x^4 u(x,y) + 2\partial_x^2\partial_y^2 u(x,y) + \partial_y^4 u(x,y)$$

(Note, that we solve in 2d, not in 3d. Since the thickness is assumed to be constant even after deformation, the appearing terms in the third space-dimension of the stress tensor can be neglected.)

- **(a)** Derive a symmetric weak formulation of the plate equation. Investigate the boundary terms and deduce "natural" boundary conditions in this situation (meaning, that they appear by applying partial integration).
- **(b)** In case of a second order pde with homogeneous Dirichlet boundary condition, we enforce this condition by constraining the space $H^1(\Omega)$ to $H^1_0(\Omega)$. What kind of boundary conditions can you set here in a similar way?
- **(c)** **Bonus:** Discuss unique existence of a solution for a boundary condition of your choice.

Problem 4.2: Galerkin equations

Starting point is the one-dimensional problem

$$-u'' + u = f \quad \text{in } \Omega = (0,1),$$

for the space $V = H^1_0(\Omega)$.

We consider the equidistant mesh

$$x_j = jh, \quad j = 0, \dots, N, \quad \text{with } h = \frac{1}{N}$$

on the interval Ω with N mesh cells $I_j = (x_{j-1}, x_j)$. The finite-dimensional subspace V_h is now the span of piecewise linear functions

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h}, & \text{if } x \in (x_{j-1}, x_j], \\ \frac{x_{j+1} - x}{h}, & \text{if } x \in (x_j, x_{j+1}), \\ 0, & \text{otherwise.} \end{cases}$$

where $j = 1, \dots, N-1$.

- **(a)** Sketch the domain Ω with its subdivision and a reasonable number of functions φ_j.
- **(b)** Argue that the space V_h contains exactly all functions which are linear on each cell I_j, continuous on Ω and have zero boundary conditions.
- **(c)** Set up the Galerkin equations.
- **(d)** Compute the 2×2 cell matrices A_k and cell vectors b_K.
- **(e)** Assemble the cell matrices and cell vectors into the global matrix A and the global vector b.
- **(f)** **Bonus (2 points):** Calculate the solution of Galerkin equations for the right hand side $f(x) = 1$ and plot it for $N = 5, 10, 20, 40, 100$ in one figure.