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Homework No. 5
Numerical Methods for PDE, Winter 2016/17

Problem 5.1: Integral node functionals

A finite element on a triangle shall consist of the space of quadratic polynomials P_2 and shall utilize the node functionals $\mathcal{N}_i$ defined by

$$\mathcal{N}_i(f) = f(p^i) \qquad i = 1, 2, 3,$$
$$\mathcal{N}_i(f) = \frac{1}{|E_{i-3}|}\int_{E_{i-3}} f(x)\,\mathrm{d}s, \qquad i = 4, 5, 6.$$

Here, E_i is the edge of the triangle facing the vertex p^i, and $|E_i|$ is its measure.

(a) Show that this element is unisolvent.

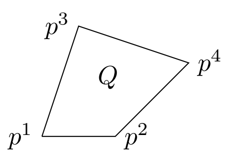
(b) Derive a basis $\{\varphi_j\}$ for P_2 such that $\mathcal{N}_i(\varphi_j) = \delta_{ij}$.

Problem 5.2: Transformation of Quadrilaterals (6 points)

Transformation from the reference square $\widehat{Q} = [0, 1]^2$ to a general quadrilateral given by vertices $p^i = (x_i, y_i)^T$ for $i = 1, \dots, 4$ can be obtained by the mapping F with

$$F(\boldsymbol{\xi}) = p^1(1-\xi)(1-\eta) + p^2\xi(1-\eta) + p^3(1-\xi)\eta + p^4\xi\eta.$$

Here, $\boldsymbol{\xi} = (\xi, \eta)^T$. The order of vertices follows the scheme



(a) Show that indeed $Q = F(\widehat{Q})$.

(b) Compute $\nabla F(\boldsymbol{\xi})$.

(c) Compute the Jacobi determinant $J(\xi)$ and show that $J(\xi) \geq 0$, if and only if the quadrilateral is convex.

(d) Compute the eigenvalues of $(\nabla F(\boldsymbol{\xi}))^T \nabla F(\boldsymbol{\xi})$ and relate them to $\|F(\boldsymbol{\xi})\|$ and $\|(F(\boldsymbol{\xi}))^{-1}\|$.

(e) What happens to $J(\xi)$, $\|F(\boldsymbol{\xi})\|$ and $\|(F(\boldsymbol{\xi}))^{-1}\|$ at p^1 if p^2 gets close to p^1?