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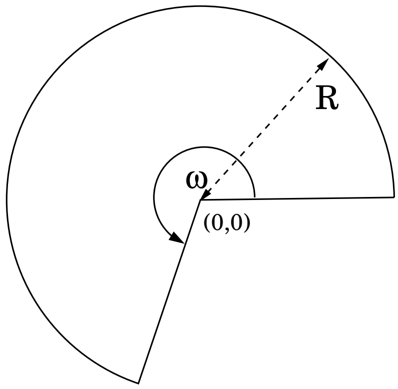
Homework No. 6
Numerical Methods for PDE, Winter 2016/17

Problem 6.1: Corner singularity

Let the domain $\Omega \subset \mathbb{R}^2$ be the sector



with radius $R = 1$ and interior angle ω. In polar coordinates, this domain is described by $r \in (0, 1)$ and $\vartheta \in (0, \omega)$.

(a) Verify: The Laplace equation in polar coordinates is

$$-\frac{\partial^2}{\partial r^2}u(r,\vartheta) - \frac{1}{r}\frac{\partial}{\partial r}u(r,\vartheta) - \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}u(r,\vartheta) = f(r,\vartheta).$$

(b) Verify that the function

$$u(r,\vartheta) = r^{\frac{\pi}{\omega}} \sin\left(\tfrac{\pi}{\omega}\vartheta\right)$$

solves the Laplace equation with zero boundary values on the legs of the angle and smooth boundary values $\sin\left(\frac{\pi}{\omega}\vartheta\right)$ on the circumference.

(c) Show that $u \notin W^{2,2}(\Omega)$ if $\omega > \pi$. **Hint:** It is sufficient to consider the derivative $\partial_{rr}u$.

(d) Show that on a triangle of size h containing the origin, this function cannot be approximated by linear functions better than

$$|u - u_h|_1 \lesssim h^{\frac{\pi}{\omega}}.$$

Here, the operator "$\lesssim$" means: there is a positive constant c independent of h (but in this case depending on u) such that $|u - u_h|_1 \le c h^{\frac{\pi}{\omega}}$