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Homework No. 8
Numerical Methods for PDE, Winter 2016/17

Problem 8.1: H^1- and L^2-Error Estimates

Consider the following problem

$$\begin{aligned} -\Delta u + u &= f, \qquad &&\text{in } \Omega, \\ u &= 0, &&\text{on } \partial\Omega, \end{aligned}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$.

We want to use a conforming FE method with piecewise linear finite elements ($V_h \subset V$) to approximate the above problem.

(a) Formulate the variational equation and its approximation, and name the solution space V as well as the ansatz spaces V_h.

(b) Discuss unique existence of the exact solution $u \in V$ as well as of the discrete solutions $u_h \in V_h$.

(c) Derive the energy error estimate

$$\|\nabla(u - u_h)\|_{L^2} \le ch\|f\|_{L^2}.$$

(d) Derive the L^2-error estimate

$$\|u - u_h\|_{L^2} \le ch^2\|f\|_{L^2}$$

by Aubin's and Nitsche's duality argument.

Hint: For a given pde in weak form "Find $u \in V$, s.t. $a(u, \varphi) = (f, \varphi)\ \forall \varphi \in V$" the dual problem is given by "Find $z \in V$, s.t. $a(v, z) = J(v)\ \forall v \in V$", where the error functional $J(\cdot)$ can be defined appropriately to the specific target.

(e) Generalize the estimates for the energy error and the L^2-error to the case of piecewise polynomial ansatz functions of polynomial degree k.