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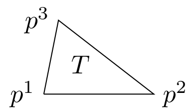
Homework No. 11
Numerical Methods for PDE, Winter 2016/17

Problem 11.1: Connection between the Shape of Triangles and the Stiffness Matrix

We discretize the Poisson problem by piecewise linear finite elements on a triangulation of the unite square. Consider an arbitrary triangle T in this triangulation.



On this triangle we consider the (parametric) linear finite element with shape functions such that $\varphi_i(p^j) = \delta_{ij}$. Remember that all derivatives involved are constant vectors and observe their directions.

(a) Show that

$$\begin{aligned} (\nabla\varphi_i, \nabla\varphi_i)_T &> 0, \qquad i = 1,2,3, \\ (\nabla\varphi_i, \nabla\varphi_j)_T &\le 0, \qquad i \neq j \end{aligned}$$

as long as all interior angles are equal or smaller than $\frac{\pi}{2}$. Use the transformation from the reference cell $\hat{T}$ and remember the formula $\cos(\alpha) = \frac{a*b}{|a||b|}$ for the angle α between the two vectors $a, b \in \mathbb{R}^2$.

(b) The entries of the global stiffness matrix are given by

$$a_{ij} = \sum_{T\in\mathcal{T}_h} (\nabla\varphi_i, \nabla\varphi_j)_T \,.$$

Conclude from the first part of the exercise that the diagonal entries of the matrix are always positive and the off-diagonal entries are smaller or equal to zero.

Problem 11.2: Error Estimate for the L^2-Projection

Let $V_h \subset H^1(\Omega)$ be the space of piecewise linear finite elements concerning a regular triangulation of $\overline{\Omega} \subset \mathbb{R}^2$. The L^2-projection $P_h\colon L^2(\Omega) \to V_h$ is defined by

$$(P_h u, v_h)_{L^2} = (u, v_h)_{L^2} \qquad \forall v_h \in V_h,$$

where $(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(\Omega)$.

(a) Proof the error estimate

$$\|u - P_h u\|_{L^2} \le ch^2 \|\nabla^2 u\|_{L^2} \tag{11.1}$$

assuming that $u \in H^2(\Omega)$, by first proving the "best-approximation-property" $\|u - P_h u\| = \inf_{v_h\in V_h} \|u - v_h\|$.

(b) What estimates can be derived under the regularity assumptions $u \in H^1(\Omega)$ and $u \in L^2(\Omega)$?

(c) How can estimate (11.1) be enhanced by using the "negative" Sobolev-norm

$$\|u - P_h u\|_{H^{-1}} := \sup_{v\in H_0^1(\Omega)} \frac{(u - P_h u, v)_{L^2}}{\|\nabla v\|_{L^2}} \le ch^? \|\nabla^2 u\|_{L^2}.$$