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Homework No. 12
Numerical Methods for PDE, Winter 2016/17

Problem 12.1: Condition Number of the Stiffness Matrix

Consider the plate equation

$$\begin{aligned} \Delta^2 u = f \qquad & \text{in } \Omega, \\ u = \partial_n u = 0 \qquad & \text{on } \partial\Omega, \end{aligned}$$

where Δ^2 denotes the biharmonic operator as definied in Problem 4.1. The problem is solved by a linear finite element method on a quasi-uniform triangular mesh. The stiffness matrix A of the plate equation is given by $a_{ij} = (\Delta\varphi_i, \Delta\varphi_j)_{L^2}$. Its maximal and minimal eigenvalue is given by λ_{max} and λ_{min}.

Compute the asympotic dependency on h of the spectral condition number

$$\text{cond}_2(A) = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}$$

of stiffness matrix A.

Problem 12.2: Iterative Solver

In practice, we are often not able to solve a linear system of equations $Ax = b$ directly by inverting matrix A or using Gauss elimination. In this case we make use of iterative solvers. Assume a contraction number ρ of an iterative solver in the form of

$$\rho = 1 - \frac{1}{\kappa}, \qquad \kappa > 1.$$

How many steps do you need to achieve an error reduction of factor ϵ?

Problem 12.3: Upwind Discretization

Consider the (hyperbolic) advection-reaction problem

$$\begin{aligned} Lu := \beta \cdot \nabla u + \gamma u = f \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \Gamma_-, \end{aligned}$$

assuming that $\gamma - (\nabla \cdot \beta)/2 \geq c > 0$. Here $\Gamma_- = \{x \in \partial\Omega : \beta \cdot n < 0\} \subset \partial\Omega$ denotes the inflow part of the boundary of Ω and n is the outward normal vector. With $H^{1,\beta}_- = \{u \in L^2(\Omega) : Lu \in L^2(\Omega), (\beta \cdot n)u = 0 \text{ on } \Gamma_-\}$ the weak formulation is given by: Find $u \in H^{1,\beta}_-$, s.t.

$$\int_\Omega (\beta \cdot \nabla u + \gamma u) v \, dx = \int_\Omega f v \, dx \qquad \forall v \in L^2(\Omega).$$

We discretize this problem by a linear dGFEM (discontinous Galerkin Finite Element Method) on a triangulation T of Ω. Therefore we define for $u_h, v_h \in V_h$

$$B(u_h, v_h) = \sum_{\kappa \in T} \left(\int_\kappa (\beta \cdot \nabla u_h + \gamma u_h) v_h \, dx - \int_{\partial_- \kappa \cap \Gamma_-} (\beta \cdot n) u_h^+ v_h^+ \, ds - \int_{\partial_- \kappa \setminus \Gamma_-} (\beta \cdot n_k) \lfloor u_h \rfloor v_h^+ \, ds \right)$$

$$F(v_h) = \sum_{\kappa \in T} \int_\kappa f v_h \, dx,$$

where $\partial_- \kappa$ denotes the inflow part of the boundary of a cell κ, $\lfloor u_h \rfloor = u_h^+ - u_h^-$ is the jump between the inner trace u_h^+ and the outer trace u_h^- of a cell and n, n_κ are the outward normal vectors of the domain Ω and the cell κ respectively.

The discrete problem is then given by: Find $u_h \in V_h = \{v_h \in L^2(\Omega) : v|_\kappa \in P_1(\kappa)\}$, s.t.

$$B(u_h, v_h) = F(v_h) \qquad \forall v_h \in V_h.$$

(a) Show consistency of the method, i.e. does the exact solution u solve the discrete problem: $B(u, v_h) = F(v_h), \forall v_h \in V_h$?

(b) Show V-ellipticity in L^2.

(c) Show boundedness in H^1.

(d) How could we get error estimates? Which steps are necessary?