Review Finite Element Methods

1. Function spaces and weak formulation
   (a) Strong formulation, minimization of energy integral and its variational equation (weak formulation)
      - integration by parts, Green's formulas
      - natural boundary conditions

   (b) Basic principle of modern analysis: write a linear PDE as a bilinear form and choose a matching function space V, such that
      i. the bilinear form is well-defined on V, that is, it is bounded (continuous)
      ii. there is a unique solution by Riesz representation or Lax-Milgram (V-ellipticity)

   (c) The weak formulation of a PDE involves integrals over Ω as inner products. Therefore, we introduce weak derivatives
      - Sobolev spaces, embedding theorems
      - Poincaré-Friedrichs inequalities
      - Trace theorems, continuity

   (d) Regularity of weak solutions
      - elliptic regularity (H^2 for right hand side in L^2)
      - corner singularities

   (e) Topology of function spaces
      - No norm-equivalence

2. Abstract Galerkin approximation
   (a) Conforming approximation, Galerkin orthogonality, Céa's lemma
      - bestapproximation
      - quasi-bestapproximation

   (b) Perturbed problems, Strang's first lemma
      - V_h-ellipticity
      - discrete operator norms!

   (c) Consistent discretization
      - DG-methods

3. Conforming finite element methods
   (a) Construction of the method

- Meshes and degrees of freedom
  - shape regular, quasi-uniform, etc.
- Shape functions and basis functions
  - continuity through node values (degrees of freedom)
  - unisolvence
- Structure of finite element integrals
  - Summing over cells
- Reference cell and transformation
- Quadrature and quadrature error

(b) Finite element spaces

- Approximation properties in Sobolev spaces
  - Bramble-Hilbert lemma
- Inverse estimates

(c) Error estimates

- a priori in H^1
- a priori in L^2
  - duality argument (Aubin–Nitsche)
- a posteriori (energy norm)
  - reliability
  - efficiency
- a posteriori (goal oriented)

4. Weak boundary conditions and DG methods

(a) Consistency and adjoint consistency

(b) discrete stability

(c) convergence estimates

5. Solving the discrete problems

(a) Richardson and conjugate gradient method

- Typical convergence estimates

(b) Preconditioned methods

(c) Multigrid methods(principles, not proofs)

- mesh and subspace hierarchies
- recursion, V-cycle, W-cycle
- smoothers