1.4.9 Problem: Verify: the first order necessary conditions of the Lagrange multiplier rule are

$$\begin{array}{lll} a(u,v)+b(v,p) & = f(v) & \forall v \in V, \\ b(u,q) & = \;\; 0 & \forall q \in Q. \end{array}$$

To this end, recall the type of objects that derivatives of linear functionals are and compute the derivatives of $\mathscr{L}$.

2.1.3 Problem: On the space $\ell_2(\mathbb{R})$ define the operator A by its eigenvalue decomposition

$$\begin{aligned} A : \ell_2(\mathbb{R}) &\to \ell_2(\mathbb{R}) \\ e_k &\mapsto \tfrac{1}{k} e_k. \end{aligned}$$

Here, $\{e_k\}$ is the orthogonal basis of unit vectors of the form

$$e_k = (0\,,\ldots,0\,,\underset{\substack{\uparrow \\ k}}{1}\,,0\,,\ldots)^T.$$

1. Show that this operator does not have a bounded inverse, albeit its eigenvalues are positive.
2. Show that the range of A is not closed in $\ell_2(\mathbb{R})$

2.1.4 Problem: Find an invertible, symmetric matrix $A \in \mathbb{R}^{2\times 2}$ and a vector $v \in \mathbb{R}^2$ such that $v^T A v = 0$ and thus the Lax-Milgram lemma is inconclusive.

2.1.17 Problem: The following statements are equivalent to the inf-sup condition (2.10):

$$\forall u \in V \; \exists w \in W \;:\; a(u,w) \geq \gamma \|u\|_V \|w\|_W \tag{3}$$

$$\forall u \in V \; \exists w \in W \;:\; \begin{cases} \|w\|_W \leq \|u\|_V \\ a(u,w) \geq \gamma \|u\|_V^2 \end{cases} \tag{4}$$

$$\forall u \in V \; \exists w \in W \;:\; \begin{cases} \gamma \|w\|_W \leq \|u\|_V \\ Aw = u \end{cases} \tag{5}$$