

Numerical Analysis of Ordinary Differential Equations

Programming Exercises

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Exercise Sheet 4

Problem P4.1 (Heat Equation)

In this exercise we want to solve the heat equation on the time-space cylinder $I \times \Omega = [0, 1] \times [0, 1]$

$$\begin{aligned}\partial_t u(t, x) &= \partial_x^2 u(t, x) & t \geq 0 \\ u(0, x) &= u_0(x)\end{aligned}\tag{P4.1}$$

We consider two cases of initial values:

1. $u_0^c(x) = \sin(\pi x)$
2. $u_0^{dc}(x) = \begin{cases} 1, & \text{for } x \in (0.3, 0.7] \\ 0, & \text{otherwise.} \end{cases}$

This PDE models the distribution of temperature in Ω over time $t \in I$. Imagine a burning candle in the area Ω . The initial temperature distribution is described by some continuous initial value U_0^c . Now the candle gets extinguished and the domain starts to cool, eventually becoming constant in Ω . The temperature distribution as a function of time since the candle is extinguished is represented by $U(t, x)$.

After discretizing (P4.1) in space (black magic from PDE-Numerics) we receive the following linear system of ODEs

$$\begin{aligned}\partial_t U(t) &= AU(t) & t \geq 0 \\ U(0) &= U_0\end{aligned}\tag{P4.2}$$

with a matrix $A \in \mathbb{R}^{(m+1) \times (m+1)}$ and a solution vector $U(t) \in \mathbb{R}^{(m+1)}$. Each of these $m + 1$ equations represent the temperature $(U(t))_i = u(t, x_i)$ in one point x_i of Ω . The stiffness-matrix A is defined by

$$A = m^2 \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

We need to discretize the initial values as well

1. $(U_0^c)_i = \sin(\pi x_i)$,
2. $(U_0^{dc})_i = \begin{cases} 1, & \text{for } x_i \in (0.3, 0.7] \\ 0, & \text{otherwise.} \end{cases}$

For equidistant steps in space, use $x_i = \frac{i-1}{m}, i = 1, \dots, m + 1$.

- a) Implement the stiffness-matrix A .
- b) Implement the Theta-method for linear ODEs.
- c) Solve the discretized heat equation (P4.2) for both initial values and $\theta = 1$. Use $m = 100$ equidistant steps in space and $n = 100$ equidistant timesteps.
- d) Plot your solution $U(t_i, x)$ in each timestep.

Remark: The solution is 2-dimensional. In Matlab you can use

```
[X,Y]=meshgrid(...)  
plot3(X,Y,U)
```

Problem P4.2 (Nonlinear ODEs)

- a) Solve the equation

$$\begin{aligned} \partial_t u(t) &= u(t)^2, & t \in [2, 2.5] \\ u(2) &= 1 \end{aligned}$$

with the Theta-method, for $\theta = 0, 0.5, 1$ and for the step-sizes $h = \frac{1}{10} \cdot 0.5^i, i = 1, \dots, 5$. Plot your results. What happens when you try to calculate the solution for $t \rightarrow 3$?

- b) Calculate convergence rates α at $t = 2.5$ for $\theta = 0.5, 0.5001, 1$ with the heuristic formula

$$\alpha = \frac{1}{\log(2)} \log \left(\frac{|y_h - y_{h/2}|}{|y_{h/2} - y_{h/4}|} \right)$$

Choose as step-sizes $h = \frac{1}{10} \cdot 0.5^i, i = 1, \dots, 10$. Explain your observations and compare them with the theoretical results shown in the lecture.