Numerical Analysis of Ordinary Differential Equations

Exercises

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Universität Heidelberg - IWR
Prof. Dr. Guido Kanschat, Dr. Dörte Jando,
Bastian Boll, Pablo Lucero

Exercise Sheet 2
Until: Mo, 30.04.2018, Noon (12:00h)

Problem 2.1 (**Order of Convergence**)
In (almost) every case the order of convergence of calculated solutions can only be determined experimentally. If we assume polynomial convergence

$$a(h) \to a, (h \to 0); \qquad a(h) - a = \mathcal{O}(h^\alpha),$$

and if the limit a is well-known, the convergence order α can be calculated

$$\alpha = \frac{1}{\log(2)} \log\left(\left|\frac{a(h) - a}{a(\frac{h}{2}) - a}\right|\right).$$

a) Recapitulate the definition of the Landau notation $f(h) = \mathcal{O}(g(h)), h \to 0$.

b) Use the formal ansatz $a(h) - a = ch^\alpha$, based on the assumption of polynomial convergence, to verify the formula.

c) In most relevant numerical experiments one does not know the limit a. How can you compute the order of convergence in that case?

Problem 2.2 (**n-body problem**)
The classical two-body problem models the motion of two constant masses m_1 and m_2 at positions x_1 and x_2 in $\mathbb{R}^3$ that only interact with each other. With the gravitational constant g it holds:

$$\begin{aligned} m_1 x_1''(t) &= -F_{21}(x_1, x_2) = -m_1 m_2 g \frac{x_1 - x_2}{\|x_1 - x_2\|^3}, \\ m_2 x_2''(t) &= -F_{12}(x_1, x_2) = -m_1 m_2 g \frac{x_2 - x_1}{\|x_1 - x_2\|^3}. \end{aligned}$$

a) Show that the momentum $p = m_1 x_1'(t) + m_2 x_2'(t)$ is independent of time t.

b) Now consider n constant masses m_i with positions x_i. Derive a system of ODEs using the superposition principle of forces.

c) Rewrite this system of second order ODEs into a system of first order ODEs.

Problem 2.3 (**Grönwall's inequality**)
Proof the differential formulation of Grönwall's inequality.
Let I denote an Intervall of the form $[a, b]$ or $[a, \infty)$, with $b > a$. Let $w(t)$ and $b(t)$ be continuous functions. Furthermore, let $w(t)$ be differentiable and satisfy the differential inequality

$$w'(t) \leqslant b(t)w(t).$$

Then w is bounded:

$$w(t) \leqslant w(a) \exp\left(\int_a^t b(s)\mathrm{d}s\right), \qquad \forall t \in I.$$