Numerical Analysis of Ordinary Differential Equations

Exercises

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Exercise Sheet 8
Until: Mon, 11.06.2018, Noon (12:00h)

Problem 8.1 (**Eigenvalues of the Laplace operator**)

The heat equation was introduced in the programming exercise sheet 4 as follows

$$\partial_t u(t,x) - \Delta u(t,x) = 0,$$

with $\Delta = \partial_{xx}$. The solution $u(t,x)$ describes the temperature distribution in time and space $(t,x) \in I \times \Omega \subset \mathbb{R} \times \mathbb{R}$.

The reformulation into a system of ODEs is given by

$$\begin{aligned} \partial_t U(t) &= -AU(t) \qquad t \geq 0 \\ U(0) &= U_0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times m}$ is a finite representation of the infinite operator $-\Delta$. If we choose $\Omega = [0,1]$ and discretize, we obtain

$$A = \frac{1}{k^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

with $k = \frac{1}{m}$. The parameter k describes the coarseness of the spatial discretization. The smaller the step size k in space, the better the resolution in space, but the larger the ODE-system to be solved.

a) Show that A is a symmetric, positive definite matrix. What can be said about its eigenvalues λ_ν based on its symmetry and positive definiteness?

b) The eigenvalues of A are given by:

$$\lambda_\nu = \frac{1}{k^2}(2 - 2\cos(\nu\pi k)), \qquad \nu = 1, \ldots, m-1.$$

Show that $\lambda_{\max} \approx \frac{1}{k^2}$. Calculate the necessary amount of timesteps to get a stable solution on the time interval $I = [0,1]$ using the explicit Euler method with $k = 10^{-3}$.

Problem 8.2 (Laplace and finite differences)

We aim to solve the Laplace equation using finite differences on the unit square $\Omega = (0,1) \times (0,1)$.

a) Write the system matrix for a discretization of 3×3 cells using Dirichlet boundary conditions.

b) Write the system matrix for a discretization of 2×2 cells using Neumann boundary conditions.

c) Let $A \in \mathbb{R}^{n \times n}$ be such that $A_{ii} > 0, A_{ij} \leq 0, i, j = 1, \dots, n, j \neq i$. Also let

$$\sum_{j \neq i, j \in I} |A_{ij}| = |A_{ii}| \qquad \forall i \in I$$

Show that A is not invertible.

d) Now use a discretization of 4×4 cells. Write the system matrix using periodic boundary value conditions. Does the problem have a unique solution?