Numerical Analysis of Ordinary Differential Equations

Exercises

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Exercise Sheet 9

The following questions are an opportunity to ensure that you understood the lecture. Please don't hand in your answers.

Problem 9.1 (**Questions I: ODE theory**)

a) Consider the following d-dimensional linear IVP:

$$u'(t) + Au(t) = b(t), \quad t \geqslant 0, \quad u(0) = u_0,$$

with a matrix $A \in \mathbb{R}^{n \times n}$ and a continuous vector-valued function $b : [0, \infty) \to \mathbb{R}^n$. Discuss existence and uniqueness of the solution $u(t)$. Is the solution bounded?

b) Which condition guarantees local existence of a solution?

c) Which condition do you need for uniqueness of a solution?

d) What is the solution of $u'(t) = \lambda u(t), t \geqslant 0$ with $u(0) = 1$?

Problem 9.2 (**Questions II: One-step methods**)

a) What is Grönwall's inequality and what is its use?

b) How is the truncation error of a one-step method $y_1 = y_0 + hF(h; t_0, y_1, y_0)$ defined?

c) How are convergence order and order of consistency defined?
How are they connected?

d) Describe the construction principle of Runge-Kutta methods within a few words.

e) Describe the structure of explicit RK methods.

f) Does there exist a RK method of order 4 with 3 stages?

g) State the definiton of the trapezoidal rule.
What is the order of the trapezoidal rule?

h) Formulate the Theta-method.

i) Describe embedded RK methods and their application.

j) Discuss the additional effort required for embedded RK methods compared to regular RK methods?

k) What is the meaning of "45" in Dormand-Prince 45?

l) What is the meaning of the acronyms DIRK and SDIRK? Describe their Butcher-tableau.

m) Describe continuous RK methods.

n) What are collocation methods?

o) What are Gauß-collocation methods?

p) How are collocation methods related to RK methods?

Problem 9.3 (**Questions III: Stability and stiffness**)

a) What is the definition of the stability region?

b) What are the stability regions of the explicit Euler method, implicit Euler method and trapezoidal rule?

c) What is the relation of the general linear IVP $u'(t) = Au(t),\ t \geqslant 0,\ u(0) = u_0$ with $u \in \mathbb{R}^n$ and the scalar model problem $u'(t) = \lambda u(t),\ t \geqslant 0,\ u(0) = 1$?

d) What is stiffness in the context of numerical analysis?

e) Is an IVP with eigenvalues $\lambda = 50 \pm \mathrm{i}80$ and $\lambda = \pm 2\mathrm{i}$ stiff?

f) When is a method considered to be A-stable?

g) Name two A-stable methods.

h) Is Dormand-Prince 45 A-stable?

i) When is a method called L-stable?

j) What is the stability function of the Theta-method?

k) What is B-stability and how is it related to A-stability?

Problem 9.4 (**Questions IV: Multistep methods**)

a) Formulate a general linear multistep method (LMM).

b) What are the first and second generating polynomials?

c) When is an LMM called stable (null-stable / D-stable)?

d) What is the connection of stability and the generating polynomials?

e) What are the conditions of convergence for an LMM?

f) What is the definition of $A(\alpha)$-stability?

Problem 9.5 (**Questions IV**)

a) Consider the equation $u'(t) = iu(t))$. What behavior can you expect, if you solve with the explicit Euler method or the implicit Euler method?

b) What is the purpose of adaptive refinement?

c) How is e^A defined? What is it used for in numerical analysis of ODEs? How does it look like for diagonalizable matrices?

d) What is the definition of order of convergence?

Problem 9.6 (**Questions V**)

a) How is a general boundary value problem defined? And how does the boundary condition simplify, if it is linear?

b) How is the Gâteaux-derivative of a function F defined?

c) How is the single shooting method motivated?

d) Describe the variational equation. In which cases could you solve it directly? And what has to be done in the other cases?

e) Where is the variational equation needed to solve BVP? How is the variational equation connected to shooting methods?

f) What is the motivation of the multiple shooting method? How is the method defined?

g) How do you modify a multiple shooting method for it to be used to solve multi-point boundary value problems?

Problem 9.7 (**Questions VI**)

a) Formulate Newton's method for the problem $f(x) = 0$, $f : \mathbb{R}^n \to \mathbb{R}^m$.

b) Describe the convergence properties of Newton's method.

c) Consider the following BVP: $u''(t) = u(t)$ with $u(0) = u(1) = 1$. How large is the matrix in Newton's method for a multiple shooting method with 5 shooting intervals?

d) What are descent methods? What is the steepest descent method?

e) What is the motivation for using a Newton's method with *step size control*?

Problem 9.8 (**Questions VII**)

a) Formulate the difference quotient for the second derivative. What is the order of that difference quotient?

b) Use the following BVP to explain the finite difference method: $u''(t) = f(t)$ in $[a, b]$ and $u(a) = u(b) = 0$.

c) What are properties of the resulting matrix in the previous question?