

$Q_k + Q_0$ -Elements in Incompressible Flows

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- 1 Introduction
- 2 The New Element Q-DG0
 - Conservation of Mass
 - Stability
 - Implementation
- 3 Numerical Results
- 4 Navier-Stokes
 - Implementation
 - Treatment of the Poisson equation
- 5 Turbulent Channel Flow
- 6 Summary

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The Stationary Stokes Equations

Find velocity \mathbf{u} and pressure p in a polyhedral domain $\Omega \subset \mathbb{R}^d$:

$$\begin{aligned} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega \end{aligned}$$

Variational formulation

Find $(\mathbf{u}, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ such that

$$\begin{aligned} \nu(\nabla \mathbf{u}, \nabla \mathbf{v})_{L^2(\Omega)} - (p, \operatorname{div} \mathbf{v})_{L^2(\Omega)} &= (\mathbf{f}, \mathbf{v})_{L^2(\Omega)} && \forall \mathbf{v} \in \mathbf{V} = H_0^1(\Omega)^d \\ (\operatorname{div} \mathbf{u}, q)_{L^2(\Omega)} &= 0 && \forall q \in Q = L_0^2(\Omega) \end{aligned}$$

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The Discretized Stokes Equations

- \mathcal{T}_h regular decomposition of Ω
- $\mathbf{V}_h \times Q_h \subset V \times Q$ finite element subspace

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$:

$$\begin{aligned} \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_{L^2(\Omega)} - (p_h, \operatorname{div} \mathbf{v}_h)_{L^2(\Omega)} &= (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)} & \forall \mathbf{v}_h \in \mathbf{V}_h \\ (\operatorname{div} \mathbf{u}_h, q_h)_{L^2(\Omega)} &= 0 & \forall q_h \in Q_h \end{aligned}$$

Remark

- Exact mass conservation is fulfilled if $\operatorname{div} \mathbf{V}_h \subset Q_h$.
- For the Taylor-Hood element just $\int_{\Omega} \operatorname{div} \mathbf{u}_h \, dx = 0$ can be expected.

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The Inf-Sup Condition

Theorem

The discretized Stokes problem has exactly one solution, if the spaces \mathbf{V}_h and Q_h satisfy the inf-sup condition

$\exists \beta > 0, \beta \neq \beta(h)$ such that

$$\inf_{\substack{q_h \in Q_h \\ q_h \neq 0}} \sup_{\substack{\mathbf{v}_h \in \mathbf{V}_h \\ \mathbf{v}_h \neq 0}} \frac{\int_{\Omega} q_h \operatorname{div} \mathbf{v}_h \, dx}{\|\mathbf{v}_h\|_{\mathbf{V}} \|q_h\|_Q} \geq \beta.$$

Remark

- The Taylor-Hood element is inf-sup stable for all $k \geq 1$.
- The same holds, if \mathcal{P} - polynomial spaces instead of Q -spaces are used.

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A New Finite Element

We seek after an element pair that

- is inf-sup stable.
- gives better mass conservation.
- is not much more “expensive” than the Taylor-Hood pair.

Idea

Augment the pressure ansatz space Q_h^k of the Taylor-Hood element by local constant functions

$$Q_h^k := \{q \in L_0^2(\Omega) : q = q_k + q_0, q_k \in C(\bar{\Omega}), \\ q_k|_K \in \mathcal{Q}_k(K), q_0|_K \in \mathcal{Q}_0(K) \quad \forall K \in \mathcal{T}_h\}.$$

Remark

- The ansatz space for the velocities remains unchanged.
- The pressure becomes discontinuous.

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Conservation of Mass

Testing with an elementwise constant function yields

$$(\operatorname{div} u_h, q_h)_{L^2(\Omega)} \quad \forall q_h \in Q_h \Rightarrow \int_K \operatorname{div} u_h \, dx = 0 \quad \forall K \in \mathcal{T}_h$$

and the element pair is locally divergence free.

Nevertheless, the solution is in general not pointwise solenoidal:

$$\operatorname{div} \mathbf{V}_h \notin Q_h$$

Stability

Theorem (A. 2013)

Let \mathcal{T}_h be a regular decomposition of Ω with flat cell faces. Then the pair

$$\mathbf{V}_h = \{\mathbf{v} \in H_0^1(\Omega)^d : v|_K \in \mathcal{Q}_{k+1}(K)^d \quad \forall K \in \mathcal{T}_h\}$$

$$\begin{aligned} Q_h = \{q \in L_0^2(\Omega) : q = q_k + q_0, q_k \in C(\bar{\Omega}), \\ q_k|_K \in \mathcal{Q}_k(K), q_0|_K \in \mathcal{Q}_0(K) \quad \forall K \in \mathcal{T}_h\} \end{aligned}$$

for $d = 2, 3$, $k \geq 1$ is inf-sup stable.

Remark

For the case with polynomial \mathcal{P}_k see [BCGG12].

Convergence

Theorem ([GR86])

If the spaces \mathbf{V}_h and Q_h satisfy the inf-sup condition, the discrete Stokes problem is well-defined and for the approximation error holds

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} + \|p - p_h\|_Q \leq C \inf_{\substack{\mathbf{v}_h \in \mathbf{V}_h \\ q_h \in Q_h}} (\|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{V}} + \|p - q_h\|_Q)$$

where the constant C is independent of h .

Convergence

Theorem ([GR86])

If the continuous solution satisfies the regularity assumption

$$\mathbf{u} \in [H^{k+2}(\Omega) \cap H_0^1(\Omega)]^d, \quad p \in H^{k+1}(\Omega) \cap L_0^2(\Omega),$$

then the convergence result

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} + \|p - p_h\|_Q \leq C_1 h^{k+1} (\|\mathbf{u}\|_{[H^{k+2}(\Omega)]^d} + \|p\|_{H^{k+1}(\Omega)})$$

holds for the discrete solution (\mathbf{u}_h, p_h) of the discrete Stokes problem. If Ω is convex, we get

$$\|\mathbf{u} - \mathbf{u}_h\|_{[L^2(\Omega)]^d} \leq C_2 h^{k+2} (\|\mathbf{u}\|_{[H^{k+2}(\Omega)]^d} + \|p\|_{H^{k+1}(\Omega)}).$$

Implementation

Difficult to find a base \Rightarrow Add the spaces FE_Q and FE_DG0 up.

```

template <int dim> double
30 TensorProductPolynomialsConst<dim>::compute_value
   (const unsigned int i, const Point<dim> &p) const
   {
       const unsigned int max_indices = this->n_tensor_pols;
       Assert (i<=max_indices, ExclnternalError());
35 // treat the regular basis functions
       if (i<max_indices)
           return this->
               TensorProductPolynomials<dim>::compute_value(i, p);
       else
40 // this is for the constant function
           return 1.;
   }

```

Listing 1: tensor_product_polynomials_const.cc

Implementation

Difficult to find a base \Rightarrow Add the spaces FE_Q and FE_DG0 up.

Problem: Globally constant functions $\phi = c$ are in both spaces.

```

440 int first_disc_dof=local_dof_indices[dofs_per_cell-1];
    constraints.add_line (first_disc_dof);

for (++cell; cell!=endc; ++cell)
{
    cell->get_dof_indices(local_dof_indices);
445 if(cell->at_boundary())
        constraints.add_entry (first_disc_dof,
                                local_dof_indices
                                [dofs_per_cell-1],-1.);
}

```

Listing 2: tests/fe/fe_q_dg0.cc

Overview

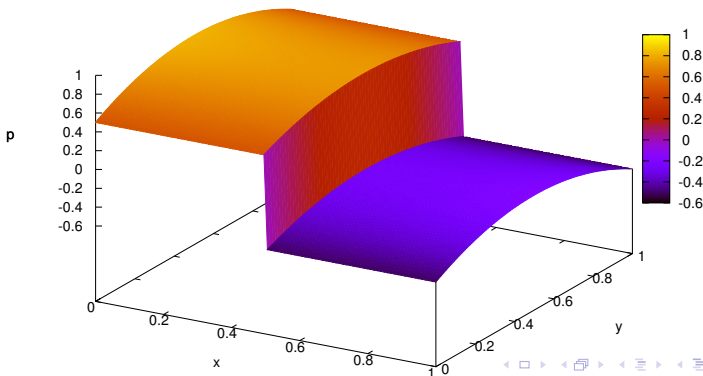
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Reference Solution

As reference solution with **discontinuous pressure** we are using

$$\mathbf{u} = (\partial_y \psi_z, -\partial_x \psi_z), \quad \psi_z = x^2(x-1)^2 y^2(y-1)^2,$$

$$p = \begin{cases} y(1-y)\exp(x-1/2)^2 + 1/2 & x \leq 1/2, \\ y(1-y)\exp(x-1/2)^2 - 1/2 & x > 1/2 \end{cases}.$$



Reference Solution

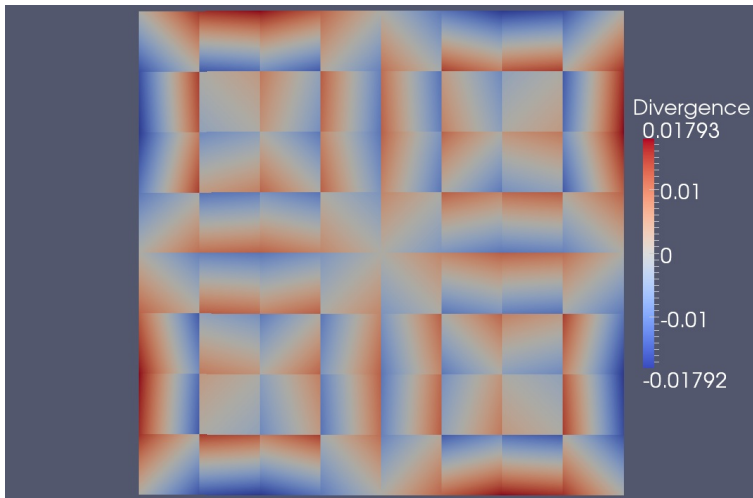
We choose the right hand side f such that the reference solution solves the Stokes problem.

$$\mathbf{f} = -\nu \Delta \mathbf{u} + \nabla p.$$

In order to take the discontinuity of the pressure into account we use the consistently modified right hand side

$$\langle \mathbf{f}, \mathbf{v} \rangle = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx + \sum_{K \in \mathcal{T}_h} \frac{1}{2} \int_{\partial K} \llbracket p \rrbracket \cdot \mathbf{v} \, dx.$$

Mass Conservation for the $Q_2/(Q_1 + Q_0)$ Pair



Convergence in the modified step-22

Table: Convergence results for the $\mathcal{Q}_2/(\mathcal{Q}_1 + \mathcal{Q}_0)$ element

$ \mathcal{T}_h $	$\ \mathbf{u} - \mathbf{u}_h\ _{L^2}$		$\ \mathbf{u} - \mathbf{u}_h\ _{H^1}$		$\ p - p_h\ _{L^2}$	
4	$1.267 \cdot 10^{-3}$	-	$1.783 \cdot 10^{-2}$	-	$2.085 \cdot 10^{-2}$	-
16	$1.714 \cdot 10^{-4}$	2.89	$4.500 \cdot 10^{-3}$	1.99	$5.231 \cdot 10^{-3}$	2.00
64	$2.151 \cdot 10^{-5}$	2.99	$1.118 \cdot 10^{-3}$	2.01	$1.304 \cdot 10^{-3}$	2.00
256	$2.687 \cdot 10^{-6}$	3.00	$2.787 \cdot 10^{-4}$	2.00	$3.258 \cdot 10^{-4}$	2.00
1024	$3.357 \cdot 10^{-7}$	3.00	$6.962 \cdot 10^{-5}$	2.00	$8.146 \cdot 10^{-5}$	2.00
4096	$4.204 \cdot 10^{-8}$	3.00	$1.740 \cdot 10^{-5}$	2.00	$2.037 \cdot 10^{-5}$	2.00
16384	$6.026 \cdot 10^{-9}$	2.80	$4.352 \cdot 10^{-6}$	2.00	$5.101 \cdot 10^{-6}$	2.00

Table: Convergence results for the $\mathcal{Q}_2/\mathcal{Q}_1$ element

⋮	⋮	⋮	⋮	⋮	⋮	⋮
16384	$2.928 \cdot 10^{-5}$	1.50	$1.678 \cdot 10^{-2}$	0.50	$3.358 \cdot 10^{-2}$	0.50

Convergence in the modified step-22

Table: Convergence results for the $\mathcal{Q}_3/(\mathcal{Q}_2 + \mathcal{Q}_0)$ element

$ \mathcal{T}_h $	$\ \mathbf{u} - \mathbf{u}_h\ _{L^2}$		$\ \mathbf{u} - \mathbf{u}_h\ _{H^1}$		$\ p - p_h\ _{L^2}$	
4	$1.002 \cdot 10^{-4}$	-	$1.968 \cdot 10^{-3}$	-	$3.744 \cdot 10^{-4}$	-
16	$6.154 \cdot 10^{-6}$	4.03	$2.351 \cdot 10^{-4}$	3.07	$3.963 \cdot 10^{-5}$	3.24
64	$3.815 \cdot 10^{-7}$	4.01	$2.898 \cdot 10^{-5}$	3.02	$5.187 \cdot 10^{-6}$	2.93
256	$2.378 \cdot 10^{-8}$	4.00	$3.609 \cdot 10^{-6}$	3.01	$6.610 \cdot 10^{-7}$	2.97
1024	$1.486 \cdot 10^{-9}$	4.00	$4.506 \cdot 10^{-7}$	3.00	$8.372 \cdot 10^{-8}$	2.98

Computational Costs in the modified step-22

Table: Comparison of running time between the Q_2/Q_1 pair and the $Q_2/(Q_1 + Q_0)$ pair

#cells	time for assembling		time for solving	
	$Q_2/(Q_1 + Q_0)$	Q_2/Q_1	$Q_2/(Q_1 + Q_0)$	Q_2/Q_1
4	$6.3 \cdot 10^{-4}$	$8.4 \cdot 10^{-4}$	$3.7 \cdot 10^{-4}$	$3.5 \cdot 10^{-4}$
16	$1.3 \cdot 10^{-3}$	$2.1 \cdot 10^{-3}$	$1.0 \cdot 10^{-3}$	$1.4 \cdot 10^{-3}$
64	$5.9 \cdot 10^{-3}$	$8.8 \cdot 10^{-3}$	$5.3 \cdot 10^{-3}$	$4.3 \cdot 10^{-3}$
256	$2.6 \cdot 10^{-2}$	$2.8 \cdot 10^{-2}$	$2.2 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$
1024	$1.2 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$	$1.0 \cdot 10^{-1}$
4096	$7.3 \cdot 10^{-1}$	$7.3 \cdot 10^{-1}$	$1.1 \cdot 10^0$	$7.3 \cdot 10^{-1}$
16384	$4.0 \cdot 10^0$	$4.0 \cdot 10^0$	$5.5 \cdot 10^0$	$3.2 \cdot 10^0$

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Instationary, Incompressible Navier-Stokes Equations

Unfiltered Navier-Stokes Equations

Find velocity \mathbf{u} and pressure p in a polyhedral domain $\Omega \subset \mathbb{R}^d$:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \times (0, T), \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega \times (0, T), \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega \times (0, T), \\ \mathbf{u}(\cdot, 0) &= \mathbf{u}_0 && \text{in } \Omega. \end{aligned}$$

⇒ additional nonlinearity and time derivative

Instationary, Incompressible Navier-Stokes Equations

LES Navier-Stokes equations

$$\begin{aligned} \frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} + \nabla \bar{p} &= \bar{\mathbf{f}} + \operatorname{div}(2(\nu + \nu_e) \mathbf{S}(\mathbf{u})) && \text{in } \Omega \times (0, T), \\ \operatorname{div} \bar{\mathbf{u}} &= 0 && \text{in } \Omega \times (0, T), \\ \bar{\mathbf{u}} &= \mathbf{0} && \text{on } \partial\Omega \times (0, T), \\ \bar{\mathbf{u}}(\cdot, 0) &= \bar{\mathbf{u}}_0 && \text{in } \Omega. \end{aligned}$$

with the symmetric strain-rate tensor

$$S_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_j} \right)$$

and the turbulence model ν_e .

Implementation

The assembled equations can be written in a matrix form as

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}.$$

A : Diffusion, Advection, Reaction

B^T : Gradient B : Divergence

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} P^{-1} \begin{pmatrix} \tilde{u} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

Use (F)GMRES with precondition matrix \tilde{P} that approximates

$$P = \begin{pmatrix} A & B^T \\ 0 & S \end{pmatrix}$$

where $S = -BA^{-1}B^T$ is the Schur complement.

Approximation to S^{-1}

In cases where reaction is dominant S^{-1} can be approximated by

$$S^{-1} = (-BA^{-1}B^T)^{-1} \approx -\Delta^{-1}$$

- **Poisson problem** with homogeneous Neumann boundary conditions
- discontinuous pressure ansatz space

⇒ Symmetric Interior Penalty Galerkin Method

SIPG - Symmetric Interior Penalty Galerkin Method

For the Poisson problem with homogeneous Neumann boundary conditions we get

$$\begin{aligned} & \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Gamma_I} \alpha \llbracket u \rrbracket \cdot \llbracket v \rrbracket - \{\{ \nabla u \} \} \cdot \llbracket v \rrbracket - \llbracket u \rrbracket \cdot \{\{ \nabla v \} \} \, dx \\ & = \int_{\Omega} f \cdot v \, dx. \end{aligned}$$

Jump operator

$$\llbracket q \rrbracket = q^+ \mathbf{n}^+ + q^- \mathbf{n}^-$$

averaging operator

$$\{\{ q \} \} = \frac{1}{2}(q^+ + q^-)$$

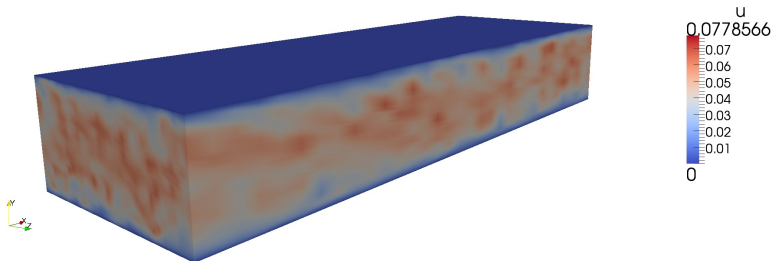
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Turbulent Channel Flow

Turbulent Channel Flow

- Flow between two infinitely extended plates
(x streamline, y anisotropic height, z width)
- $\Omega = 2\pi \times 2 \times \frac{4}{3}\pi$
- Random distortion of initial velocity (Reichelt's Law)
- $f = (0, 0, 0)$, $Re_\tau = 180$, $\nu_e(\mathbf{v}) \sim \frac{|\text{tr}(\mathbf{S}^3(\mathbf{v}))|}{\text{tr}(\mathbf{S}^2(\mathbf{v}))}$

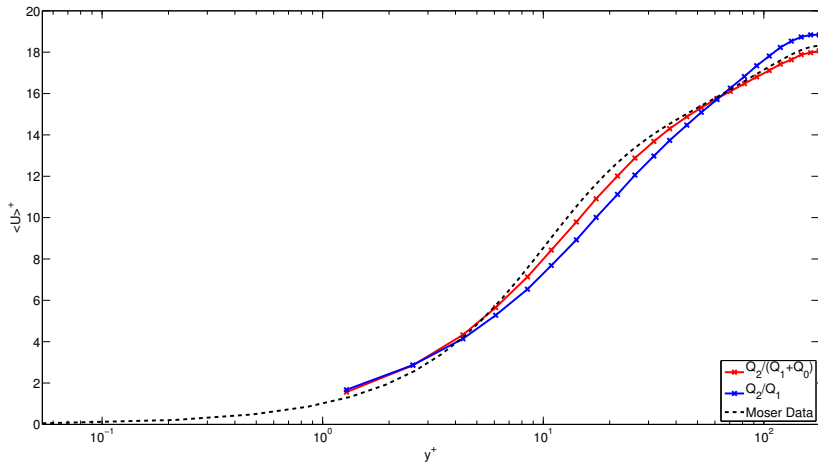


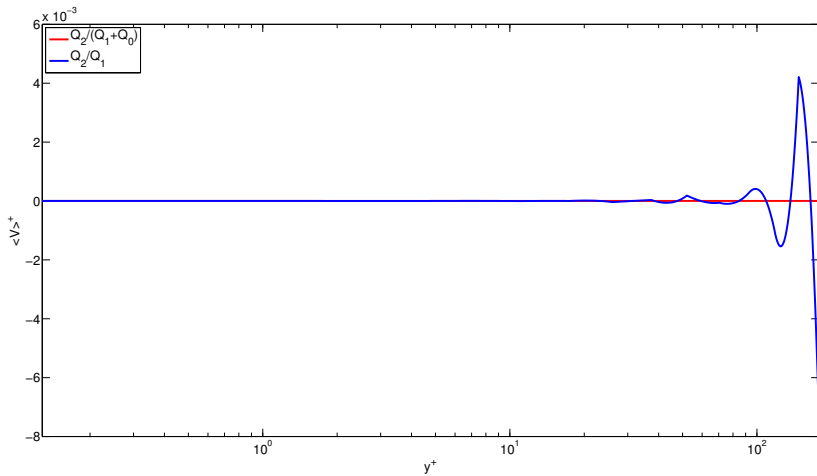
Characteristic Values

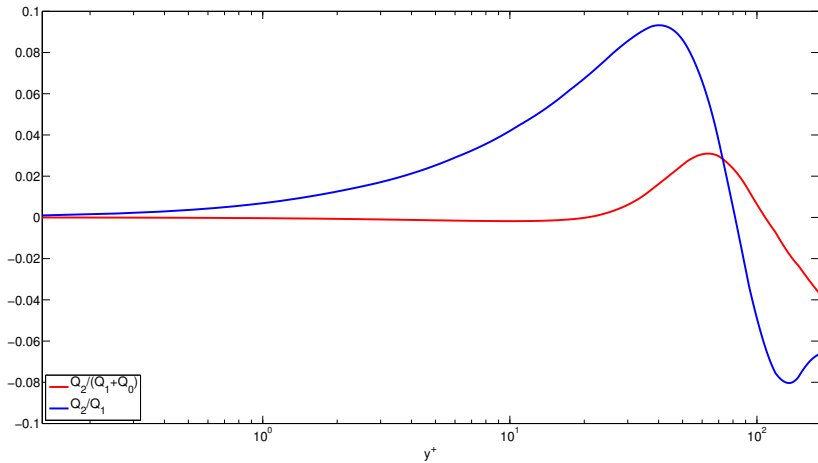
Characteristic Values

- mean value $\langle u \rangle$ averaged in time and space
- Reynolds decomposition $u = \langle u \rangle + u'$
- viscous length $y^+ = \nu \frac{\partial \langle u \rangle}{\partial y} \Big|_{y=0} y$
- $u_\tau = \sqrt{\nu \frac{\partial \langle u \rangle}{\partial y} \Big|_{y=0}}$
- $\langle u \rangle^+ = \frac{\langle u \rangle}{u_\tau}$

Reference data from [MKM99]

$\langle u \rangle^+$ 

$\langle v \rangle^+$ 

$\langle w \rangle^+$ 

Choice of the Preconditioner for the Poisson-Problem

Table: Average time and number of iterations

Preconditioner	$6 \times 6 \times 6$		$12 \times 12 \times 12$	
	iters.	time	iters.	time
Identity	170	0.08 s	490	2.4 s
BoomerAMG	90	0.30 s	340	19.0 s
Jacobi	150	0.07 s	470	2.3 s
BlockJacobi(ILU)	80	0.07 s	200	2.1 s
BlockJacobi(ICC)	100	0.08 s	130	1.8 s
ParaSails	50	0.04 s	260	1.6 s

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Summary

Results

- The Q-DG0 element is inf-sup stable for tensor product polynomials,
- The same convergence results as for the Taylor-Hood element,
- Improved approximation to the mean profile of a turbulent channel flow by local mass conservation.

Challenge

- Choice of the preconditioner in the inner solver of the Navier-Stokes problem

References



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Vielen Dank für die Aufmerksamkeit!

