Short Course on Partial Differential Equations with deal.II

Mixed Elements

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Consider the stationary Stokes problem

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}$$

which serves as a model for

- creeping flow
- incompressible fluid
- stationary flow

Mixed Finite Elements - Step-22

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- creeping flow
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Solution components

- velocity $\boldsymbol{u}$
- pressure p

Weak formulation

$$\begin{aligned}(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) &= \boldsymbol{f}, \boldsymbol{v}\\ (\nabla \cdot \boldsymbol{u}, q) &= 0\end{aligned}$$

Sattle point problem

$$\begin{pmatrix} A & -B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}$$

with

- A=-Δ
- B= ∇

Mixed Finite Elements - Step-22

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Discretization

$$\begin{aligned}(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (\nabla \cdot v_h, p_h) &= (\boldsymbol{f}, v_h)\\ (\nabla \cdot \boldsymbol{u}_h, q_h) &= 0\end{aligned}$$

Sattle point problem

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with

- A=-Δ
- B= ∇

Test problem

- $\Omega = (-5, 5) \times (-1, 1)$
- $u|_{y=\pm 1} = \mathbf{0}$
- $u_x = u_x(y)$
- $u_y = \mathbf{0}$

This means for the Stokes problem

$$\begin{aligned} (-\partial_{xx} - \partial_{yy})u_x + \partial_x p &= 0 \\ (-\partial_{xx} - \partial_{yy})u_y + \partial_y p &= 0 \\ \partial_x u_x + \partial_y u_y &= 0 \end{aligned}$$

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$$\begin{aligned} -\partial_{yy} u_x + \partial_x p &= 0 \\ \partial_y p &= 0 \\ \partial_x u_x &= 0 \end{aligned}$$

Mixed Finite Elements - Step-22

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This means for the Stokes problem

$$\begin{aligned} \partial_{yy} u_x = \partial_x p &= C \\ \partial_y p &= 0 \\ \partial_x u_x &= 0 \end{aligned}$$

Test problem

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- $u|_{y=\pm 1} = \mathbf{0}$
- $u_x = u_x(y)$
- $u_y = \mathbf{0}$

This means for the Stokes problem

$$\begin{aligned} u_x &= Cy^2 + ay + b \\ u_y &= 0 \\ p &= Cx + d \end{aligned}$$

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- $u|_{y=\pm 1} = \mathbf{0}$
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This means for the Stokes problem

$$\begin{aligned} u_x &= C(y^2 - 1) \\ u_y &= 0 \\ p &= Cx + d \end{aligned}$$

Prescribe $u_x|_{x=-5} = (1 - y^2) \Rightarrow C = -1$.

Remarks on Solver

Sattle point problem

$$\begin{pmatrix} A & -B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}$$

The first equation can be expressed as $\boldsymbol{u} = A^{-1}(\boldsymbol{f} - Bp)$. Hence,

$$B^T A^{-1} B p = -B^T A^{-1} \boldsymbol{f}$$

We need only A^{-1}

Remarks on Solver

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$$\underbrace{B^T A^{-1} B}_{\approx I} p = -B^T A^{-1} \boldsymbol{f}$$

Use

- Use UMFPACK or ILU for A
- a mass-preconditioned CG solver for $B^T A^{-1} B$