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CONVERGENCE ANALYSIS OF AN ADAPTIVE INTERIOR PENALTY DISCONTINUOUS GALERKIN METHOD*

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Abstract. We study the convergence of an adaptive interior penalty discontinuous Galerkin (IPDG) method for a two-dimensional model second order elliptic boundary value problem. Based on a residual-type a posteriori error estimator, we prove that after each refinement step of the adaptive scheme we achieve a guaranteed reduction of the global discretization error in the mesh-dependent energy norm associated with the IPDG method. In contrast to recent work on adaptive IPDG methods [O. Karakashian and F. Pascal, *Convergence of Adaptive Discontinuous Galerkin Approximations of Second-order Elliptic Problems*, preprint, University of Tennessee, Knoxville, TN, 2007], the convergence analysis does not require multiple interior nodes for refined elements of the triangulation. In fact, it will be shown that bisection of the elements is sufficient. The main ingredients of the proof of the error reduction property are the reliability and a perturbed discrete local efficiency of the estimator, a bulk criterion that takes care of a proper selection of edges and elements for refinement, and a perturbed Galerkin orthogonality property with respect to the energy inner product. The results of numerical experiments are given to illustrate the performance of the adaptive method.

Key words. discontinuous Galerkin, adaptive methods, interior penalty, error estimates

AMS subject classifications. 65N30, 65N50

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1. Introduction.

During the past decade, discontinuous Galerkin (DG) methods have emerged as a powerful algorithmic tool in the numerical solution of boundary and initial boundary value problems for partial differential equations (PDE) (cf., e.g., [15, 17] and the references therein). For second order elliptic problems, one may distinguish between primal schemes and mixed methods. Primal schemes rely on augmenting the elliptic operator by an appropriate penalization of the discontinuous nodal shape functions. On the other hand, in mixed methods the second order PDE is reformulated as a system of first order PDEs for which suitable numerical fluxes are designed. The most prominent primal schemes are interior penalty discontinuous Galerkin (IPDG) methods, whereas a widely used class of mixed techniques is given by the local discontinuous Galerkin (LDG) methods. Both IPDG and LDG methods have been intensively studied with regard to an a priori error analysis in terms of error estimates for the global discretization error (see, e.g., [2, 3, 12, 26]).

The a posteriori analysis of finite element methods (FEM) is in some state of maturity, as documented by a series of monographs that have been published in recent years [1, 4, 6, 19, 32, 37]. As far as DG methods are concerned, a posteriori error estimators have been developed and analyzed for elliptic problems in H^1 in [7, 25, 27, 33, 34], for elliptic problems in $H(curl)$ in [22, 23], and for the Stokes problem in [24].

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In this paper, we will be concerned with a convergence analysis of an adaptive IPDG method in the sense that for a two-dimensional (2D) second order elliptic model problem we will prove guaranteed error reduction with respect to the mesh-dependent energy norm. We note that for standard conforming P1 approximations of elliptic problems the convergence analysis of adaptive finite element methods (AFEM) has been initiated in [5] and further studied in [18, 29, 30, 31], whereas the issue of optimal order of convergence has been addressed in [8] and [36]. Nonstandard finite element techniques such as mixed and nonconforming methods and edge element discretizations of Maxwell's equations have been recently investigated in [9, 10, 11]. In the recent paper [28], a convergence analysis of symmetric IPDG methods has been provided. In contrast to [28], our analysis does not require multiple interior nodes for refined elements of the triangulation. In fact, we show that it suffices to refine by bisection.

The paper is organized as follows: In section 2, we briefly introduce the IPDG method. Section 3 describes the adaptive loop consisting of the basic steps SOLVE, ESTIMATE, MARK, and REFINE and states the main convergence result. Section 4 recalls the reliability of the estimator from [27] and establishes a perturbed discrete local efficiency, whereas section 5 is devoted to the proof of the error reduction property. Finally, section 6 contains a documentation of the results of numerical experiments that illustrate the performance of the adaptive IPDG (AIPDG).

2. The IPDG method.

We assume $\Omega \subset \mathbb{R}^2$ to be a bounded, polygonal domain with boundary $\Gamma = \partial\Omega, \Gamma = \Gamma_D \cup \Gamma_N, \Gamma_D \cap \Gamma_N = \emptyset$. We adopt standard notation from Sobolev space theory and refer to $(\cdot,\cdot)_{k,D}$ and $\|\cdot\|_{k,D}, k \in \mathbb{N}_0, D \subseteq \Omega$, as the $H^k(D)$-inner product and associated norm, respectively.

As a model problem, for given $f \in L^2(\Omega), u^D \in H^{1/2}(\Gamma_D), u^N \in L^2(\Gamma_N)$, we consider Poisson's equation with inhomogeneous Dirichlet and Neumann boundary data

$$-\Delta u = f \quad \text{in } \Omega, \tag{2.1a}$$

$$u = u^D \quad \text{on } \Gamma_D, \tag{2.1b}$$

$$\partial_{n_{\Gamma_N}} u = u^N \quad \text{on } \Gamma_N, \tag{2.1c}$$

whose variational formulation amounts to the computation of a solution $u \in V := \{v \in H^1(\Omega) \mid v|_{\Gamma_D} = u^D\}$ such that

$$a(u,v) = (f,v)_\Omega + \langle u^N, v\rangle_{\Gamma_N}, \quad v \in H^1_{0,\Gamma_D}(\Omega), \tag{2.2}$$

where $a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx$.

For the DG approximation of (2.2), we further assume that $\mathcal{T}_H(\Omega)$ is a simplicial triangulation of Ω which aligns with Γ_D, Γ_N on the boundary Γ. For $D \subseteq \overline{\Omega}$, we denote by $|D|$ the volume of D and by $\Pi_p(D), p \in \mathbb{N}_0$, the linear space of polynomials of degree p on D, and we refer to $\mathcal{N}_H(D), \mathcal{E}_H(D)$, and $\mathcal{T}_H(D)$ as the sets of vertices, edges, and elements, respectively, in D. For $T \in \mathcal{T}_H(\Omega)$, h_T stands for the diameter of T, whereas for $E \in \mathcal{E}_H(\overline{\Omega})$, we denote by h_E the length of E. Moreover, for an interior edge $E \in \mathcal{E}_H(\Omega)$ such that $E = T_+ \cap T_-, T_\pm \in \mathcal{T}_H(\Omega)$, we refer to $\omega_E := T_+ \cup T_-$ as the patch formed by the union of the elements sharing E as a common edge. Finally, for a function $g \in L^2(D), D \subset \bar{\Omega}$, the quantity $\hat{g}_D$ stands for the integral mean of g with respect to D, i.e., $\hat{g}_D := |D|^{-1} \int_D g dx$.

We define the product space $V_H := \prod_{T\in\mathcal{T}_H(\Omega)} \Pi_p(T), p \in \mathbb{N}$, and introduce the bilinear form $a_H(\cdot,\cdot) : V_H \times V_H \to \mathbb{R}$ according to

$$a_H(u_H, v_H) := \sum_{T\in\mathcal{T}_H(\Omega)} (\nabla u_H, \nabla v_H)_T - \sum_{E\in\mathcal{E}_H(\overline{\Omega})} \Big((\{\partial_{n_E} u_H\}, [v_H])_E + ([u_H]_E, \{\partial_{n_E} v_H\})_E \Big) + \alpha \sum_{E\in\mathcal{E}_H(\overline{\Omega})} h_E^{-1} \, ([u_H]_E, [v_H]_E)_E, \tag{2.3}$$

where the normal vector on E points from T_+ to T_- and with $v_H^\pm := v_H|_{T_\pm}$ on E,

$$\begin{aligned} [v_H]_E &:= v_H^+ - v_H^-, && E \in \mathcal{E}_H(\Omega),\\ [v_H]_E &:= v_H|_E, && E \in \mathcal{E}_H(\Gamma),\\ \{v_H\}_E &:= \tfrac{1}{2}\left(v_H^+ + v_H^-\right), && E \in \mathcal{E}_H(\Omega),\\ \{v_H\}_E &:= v_H|_E, && E \in \mathcal{E}_H(\Gamma), \end{aligned}$$

and $\alpha > 0$ stands for a properly chosen penalization parameter.

Then, the interior penalty method in its symmetric formulation amounts to the computation of $u_H \in V_H$ such that

$$a_H(u_H, v_H) \;=\; \ell(v_H), \quad v_H \in V_H, \tag{2.4}$$

where

$$\ell(v_H) := (f, v_H)_\Omega + \left(u^N, v_H\right)_{\Gamma_N} - \sum_{E\subset\Gamma_D} \left(u^D, \partial_n v_H - \alpha h_E^{-1} v_H\right)_E. \tag{2.5}$$

On V_H, we introduce the mesh-dependent H^1-norm

$$\|v_H\|_{1,H,\Omega} \;:=\; \left(\sum_{T\in\mathcal{T}_H(\Omega)} \|\nabla v_H\|_T^2 + \sum_{E\in\mathcal{E}_H(\overline{\Omega})} (h_E \|\{\partial_{n_E} v_H\}\|_E^2 + h_E^{-1} \|[v_H]\|_E^2) \right)^{1/2}. \tag{2.6}$$

As has been shown in [27], the bilinear form $a_H(\cdot,\cdot)$ is bounded

$$|a_H(u_H, v_H)| \;\le\; (1+\alpha)\, \|u_H\|_{1,H,\Omega} \|v_H\|_{1,H,\Omega}, \quad u_H, v_H \in V_H, \tag{2.7}$$

and for sufficiently large α coercive with respect to the $\|\cdot\|_{1,H,\Omega}$-norm, i.e., there exist positive constants α_{min} and γ such that for $\alpha \ge \alpha_{\min}$

$$|a_H(v_H, v_H)| \;\ge\; \gamma\, \|v_H\|_{1,H,\Omega}^2, \quad v_H \in V_H. \tag{2.8}$$

It follows from (2.7) and (2.8) that for $\alpha \ge \alpha_{\min}$ the IPDG (2.4) admits a unique solution $u_H \in V_H$. Moreover, for such α the mesh-dependent energy norm

$$|||v_H|||_{H,\Omega} \;:=\; a_H(v_H, v_H)^{1/2}, \quad v_H \in V_H, \tag{2.9}$$

is equivalent to the $\|\cdot\|_{1,H,\Omega}$-norm

$$\gamma \, \|v_H\|^2_{1,H,\Omega} \;\le\; |||v_H|||^2_{H,\Omega} \;\le\; (1+\alpha)\, \|v_H\|^2_{1,H,\Omega}, \quad v_H \in V_H. \tag{2.10}$$

For a subset $D_H \subset \mathcal{T}_H(\Omega)$ of the triangulation, $\|\cdot\|_{1,H,D_H}$ and $|||\cdot|||_{H,D_H}$ are defined analogously.

Remark 2.1. We have chosen the 2D model problem (2.1a)–(2.1c) to focus on the essential ingredients for the proof of the error reduction property and not to overload the convergence analysis with too many technicalities. Using the tools from [29], we think that the results can be extended to more general elliptic differential operators, thus including advection-diffusion problems. We further believe that the ideas presented in this paper can be also adopted to hybridized DG methods [16] where the number of degrees of freedom is comparable to standard finite element discretizations.

3. The adaptive loop and the main convergence result. An adaptive FEM for the IPDG (2.4) consists of successive loops of the following sequence:

$$\text{SOLVE} \;\rightarrow\; \text{ESTIMATE} \;\rightarrow\; \text{MARK} \;\rightarrow \text{REFINE}. \tag{3.1}$$

Here, SOLVE stands for the numerical solution of (2.4) with respect to the given triangulation $\mathcal{T}_H(\Omega)$. We remark that for this purpose efficient preconditioned iterative solvers have been developed, analyzed, and implemented (cf., e.g., [20, 21, 25]).

The following residual-type a posteriori error estimator η_H has been introduced and analyzed in [27]:

$$\eta_H^2 \;:=\; \sum_{T \in \mathcal{T}_H(\Omega)} \eta_T^2 \;+\; \sum_{E \in \mathcal{E}_H(\Omega)} \eta_E^2. \tag{3.2}$$

Here, η_T stands for the element residual

$$\eta_T \;:=\; h_T \, \|f + \Delta u_H\|_T, \quad T \in \mathcal{T}_H(\Omega). \tag{3.3}$$

On the other hand, η_E summarizes the edge residuals

$$\eta_E^2 \;:=\; \eta_{E,1}^2 + \eta_{E,2}^2 + \eta_{E,N}^2 + \eta_{E,D}^2 \tag{3.4}$$

given by

$$\eta_{E,1} := \; h_E^{1/2} \, \|[\partial_{n_E} u_H]\|_E, \qquad E \in \mathcal{E}_H(\Omega), \tag{3.5a}$$

$$\eta_{E,2} := \; h_E^{-1/2} \, \|[u_H]\|_E, \qquad E \in \mathcal{E}_H(\Omega), \tag{3.5b}$$

$$\eta_{E,N} := \; h_E^{1/2} \, \|u^N - \partial_{n_E} u_H\|_E, \qquad E \in \mathcal{E}_H(\Gamma_N), \tag{3.5c}$$

$$\eta_{E,D} := \; h_E^{-1/2} \, \|u^D - u_H\|_E, \qquad E \in \mathcal{E}_H(\Gamma_D). \tag{3.5d}$$

The convergence analysis further invokes the data oscillations

$$\mathrm{osc}_H^2 \;:=\; \mathrm{osc}_H^2(f) + \mathrm{osc}_H^2\left(u^D\right) + \mathrm{osc}_H^2\left(u^N\right), \tag{3.6}$$

where

$$\mathrm{osc}_H^2(f) := \sum_{T\in\mathcal{T}_H(\Omega)} \mathrm{osc}_T^2(f), \tag{3.7a}$$

$$\mathrm{osc}_T(f) := h_T\,\|f - \hat{f}_T\|_T,$$

$$\mathrm{osc}_H^2\left(u^D\right) := \sum_{E\in\mathcal{E}_H(\Gamma_D)} \mathrm{osc}_E^2\left(u^D\right), \tag{3.7b}$$

$$\mathrm{osc}_E\left(u^D\right) := h_E^{-1/2}\,\|u^D - \hat{u}_E^D\|_E,$$

$$\mathrm{osc}_H^2\left(u^N\right) := \sum_{E\in\mathcal{E}_H(\Gamma_N)} \mathrm{osc}_E^2\left(u^N\right), \tag{3.7c}$$

$$\mathrm{osc}_E\left(u^N\right) := h_E^{1/2}\|u^N - \hat{u}_E^N\|_E.$$

In the step MARK of the adaptive loop, given a universal constant $0 < \Theta \leq 1$, we choose subsets $\mathcal{M}_T \subset \mathcal{T}_H(\Omega)$ and $\mathcal{M}_E \subset \mathcal{M}_H(\bar{\Omega})$ such that the following bulk criterion is satisfied:

$$\Theta \sum_{T\in\mathcal{T}_H(\Omega)} \eta_T^2 \leq \sum_{T\in\mathcal{M}_T} \eta_T^2, \tag{3.8a}$$

$$\Theta \sum_{E\in\mathcal{E}_H(\bar{\Omega})} \eta_E^2 \leq \sum_{E\in\mathcal{M}_E} \eta_E^2. \tag{3.8b}$$

The bulk criterion can be realized by a greedy algorithm.

As far as the data oscillations are concerned, for simplicity we assume that the set $\mathcal{M}_T$ selected by (3.8a) is already rich enough such that there exists a constant $0 \leq \rho_2 < 1$ such that

$$\mathrm{osc}_h^2 \;\leq\; \rho_2\;\mathrm{osc}_H^2\,. \tag{3.9}$$

We note that the data oscillations may be included in the bulk criterion as well to guarantee (3.9). We refer to [30, 31] for details.

The refinement strategy in the final step REFINE of the adaptive loop is as follows: If an element $T \in \mathcal{T}_H(\Omega)$ has been marked for refinement, it will be refined by longest edge bisection. If an edge $E \in \mathcal{E}_H(\Omega), E = T^+\cap T^-$ (resp., $E \in \mathcal{T}_H(\Gamma), E = \partial T \cap \Gamma$) has been marked, the triangles $T^\pm$ (resp., the triangle T) will be refined by bisection. We note that this refinement is different from that used in [28] where the refinement of a triangle requires multiple interior nodes based on subsequent regular refinements.

The main result of this paper is a guaranteed error reduction of the global discretization error measured in the mesh-dependent energy norm associated with the IPDG method.

THEOREM 3.1. *Let $u \in V$ be the solution of* (2.2)*, and suppose that $u_H \in V_H$ and $u_h \in V_h$ are the solutions of IPDG* (2.4) *with respect to the triangulation $\mathcal{T}_H(\Omega)$ and the adaptively refined triangulation $\mathcal{T}_h(\Omega)$ generated according to the refinement rules described before. Assume that* (3.9) *holds true. Then, for sufficiently large penalization parameter α there exist positive constants $\varrho_1 < 1$ and C which depend* *only on* α, Θ *and the shape regularity of the triangulations such that for* $e_H := u - u_H$ *and* $e_h := u - u_h$ *there holds*

$$\begin{pmatrix} a_h(e_h, e_h) \\ \text{osc}_h^2 \end{pmatrix} \leq \begin{pmatrix} \rho_1 & C \\ 0 & \rho_2 \end{pmatrix} \begin{pmatrix} a_H(e_H, e_H) \\ \text{osc}_H^2 \end{pmatrix}. \tag{3.10}$$

The proof of Theorem 3.1 will be given in section 5 based on the reliability and a perturbed discrete local efficiency of the estimator (3.2), which will be studied in the following section.

4. Reliability and perturbed discrete local efficiency. The reliability of the residual-type a posteriori error estimator (3.2) has been established in [27] using standard techniques from AFEM [37]. Here, we prove that it is also locally efficient in a relaxed way. We will derive the main lemmas for the case of the newest edge bisection [13, 14, 35].

THEOREM 4.1. *Let* $u \in V$ *and* $u_H \in V_H$ *be the solution of* (2.2) *and its IPDG approximation* (2.4), *and let* η_H *and* osc_H *be the residual error estimator and the data oscillations as given by* (3.2) *and* (3.6), *respectively. Then, for* $e_H := u - u_H$ *there holds*

$$a_H(e_H, e_H) \lesssim \eta_H^2. \tag{4.1}$$

Discrete local efficiency means that up to data oscillations the local contributions of the estimator can be bounded from above by the energy norm of the difference between the fine mesh and coarse mesh approximations on a refined triangle and the patch ω_E associated with a refined edge, respectively [18, 30]. In the framework of the IPDG approximations under consideration, we can prove only a perturbed discrete local efficiency in the sense that the upper bounds involve additional quantities in terms of the fine mesh approximation. In particular, the following result holds true.

THEOREM 4.2. *Let* $u \in V$ *and* $u_H \in V_H, u_h \in V_h$, *be the solution of* (2.2) *and its IPDG approximations* (2.4) *with respect to* $\mathcal{T}_H(\Omega)$ *and* $\mathcal{T}_h(\Omega)$, *respectively. Moreover, let* η_H *and* osc_H *be the residual error estimator* (3.2) *and the data oscillations* (3.6), *respectively. Then, there holds*

$$\begin{aligned} \sum_{T\in\mathcal{M}_T} \eta_T^2 + \sum_{E\in\mathcal{M}_E} \eta_E^2 \;\lesssim\; & a_h(u_h - u_H, u_h - u_H) \\ & + \alpha \sum_{E'\in\mathcal{E}_h(\mathcal{M}_E\setminus\Gamma_D)} h_{E'}^{-1} \|[u_h]\|_{E'}^2 \\ & + \alpha \sum_{E'\in\mathcal{E}_h(\mathcal{M}_E\cap\Gamma_D)} h_{E'}^{-1} \left\|u^D - u_h\right\|_{E'}^2 + \text{osc}_H^2 . \end{aligned} \tag{4.2}$$

Proof. The proof of (4.2) follows by collecting the estimates from the subsequent series of lemmas. ∎

LEMMA 4.3. *Let* $T \in \mathcal{T}_H(\Omega)$ *be a refined triangle such that* $T = T_+ \cup T_-, T_\pm \in \mathcal{T}_h(\Omega)$. *Then, there holds*

$$\begin{aligned} h_T^2 \|f + \Delta u_H\|_T^2 \lesssim\; & a_h|_T(u_h - u_H, u_h - u_H) + \text{osc}_T^2(f) \\ & + \alpha \sum_{E\in\mathcal{E}_H(\partial T\cap\Omega)} \eta_{E,2}^2 + \alpha \sum_{E\in\mathcal{E}_H(\partial T\cap\Gamma_D)} \eta_{E,D}^2 + \sum_{E\in\mathcal{E}_H(\partial T\cap\Gamma_N)} osc_E^2\left(u^N\right) . \end{aligned} \tag{4.3}$$

Proof. We denote by $CR_p(\Omega;\mathcal{T}_h(\Omega)), p \in \mathbb{N}$, the nonconforming Crouzeix–Raviart finite element space, where $v_h|_{T'} \in \Pi_p(T'), T' \in \mathcal{T}_h(\Omega)$, is uniquely determined by the degrees of freedom

$$\int_E v_h q_E \, ds, \qquad q_E \in \Pi_{p-1}(E), \; E \in \mathcal{E}_h(T'),$$
$$\int_T v_h q_{T'} \, dx, \qquad q_{T'} \in \Pi_{p-3}(T') \quad (p \geq 3).$$

We choose $\varphi_h \in V_h$ with $\varphi_h|_{T_\pm} \in \Pi_p(T_\pm)$ and $\varphi_h|_{T'} \equiv 0, T' \in \mathcal{T}_h(\Omega) \setminus \{T\}$, such that

$$h_{T_\pm}^2 \left\| \hat{f}_T + \Delta u_H \right\|_{T_\pm}^2 = \left(\hat{f}_T + \Delta u_H, \varphi_h \right)_{T_\pm}, \tag{4.4a}$$

$$\|\varphi_h\|_{T_\pm}^2 \lesssim h_{T_\pm}^4 \left\| \hat{f}_T + \Delta u_H \right\|_{T_\pm}^2, \tag{4.4b}$$

$$(q_h, \varphi_h)_E = 0, \quad q_h \in \Pi_{p-1}(E), \quad E \in \mathcal{E}_h(\partial T). \tag{4.4c}$$

In particular, in the case $p \leq 2$ we choose φ_h as a linear combination of the basis functions associated with the interior edge $E \in \mathcal{E}_h(\mathrm{int}(T))$, whereas for $p \geq 3$ we choose φ_h as a linear combination of the basis functions associated with $\mathrm{int}(T_\pm)$. Using (4.4a), Green's formula, and setting $T_1 := T_+, T_2 := T_-$, we obtain

$$\begin{aligned} h_T^2 \left\| \hat{f}_T + \Delta u_H \right\|_T^2 &= \sum_{i=1}^{2} \left(\hat{f}_T + \Delta u_H, \varphi_h \right)_{T_i} \\ &= \sum_{i=1}^{2} \left(-(\nabla u_H, \nabla \varphi_h)_{T_i} + (f, \varphi_h)_{T_i} + \left(\hat{f}_T - f, \varphi_h \right)_{T_i} \right), \end{aligned} \tag{4.5}$$

where we have used that due to (4.4c) for $T \in \mathcal{T}_H$ there holds

$$(\partial_{n_E} u_H, \varphi_h)_E = 0, \qquad E \in \mathcal{E}_h(\partial T), \qquad p \geq 1, \tag{4.6a}$$

$$(\partial_{n_E} u_H, [\varphi_h])_E = 0, \qquad E \in \mathcal{E}_h(\mathrm{int}(T)), \qquad p \geq 1. \tag{4.6b}$$

On the other hand, φ_h is an admissible test function in the fine grid equation (2.4) whence

$$\begin{aligned} &\sum_{i=1}^{2} \Big((\nabla u_h, \nabla \varphi_h)_{T_i} - (f, \varphi_h)_{T_i} \Big) \\ &+ \sum_{E \in \mathcal{E}_h(\partial T \cap \Gamma_D)} \left(u^D, \partial_{n_E} \varphi_h - \alpha h_E^{-1} \varphi_h \right)_E \\ &- \sum_{E \in \mathcal{E}_h(\partial T \cap \Gamma_N)} \left(u^N, \varphi_h \right)_E \\ &- \sum_{E \in \mathcal{E}_h(T)} \Big((\{\partial_{n_E} u_h\}, [\varphi_h])_E + ([u_h], \{\partial_{n_E} \varphi_h\})_E \Big) \\ &+ \alpha \sum_{E \in \mathcal{E}_h(T)} h_E^{-1} \; ([u_h], [\varphi_h])_E \; = \; 0. \end{aligned} \tag{4.7}$$

Adding (4.5) and (4.7) and observing again (4.6a)–(4.6b) as well as $[u_H] = 0$ on $E \in \mathcal{E}_h(\mathrm{int}(T))$, it follows that

$$
\begin{aligned}
(4.8)\qquad & h_T^2 \ \left\|\hat{f}_T + \Delta u_H\right\|_T^2 \\
&= \sum_{i=1}^{2} \Big((\nabla(u_h - u_H), \nabla\varphi_h)_{T_i} + \Big(\hat{f}_T - f, \varphi_h\Big)_{T_i} \Big) \\
&- \sum_{E\in\mathcal{E}_h(T)} \Big((\{\partial_{n_E}(u_h - u_H)\}, [\varphi_h])_E \\
&\qquad + ([u_h - u_H], \{\partial_{n_E}\varphi_h\})_E \Big) \\
&- \sum_{E\in\mathcal{E}_h(\partial T\cap\Omega)} ([u_H], \{\partial_{n_E}\varphi_h\})_E \\
&+ \sum_{E\in\mathcal{E}_h(\partial T\cap\Gamma_D)} \big(u^D - u_H, \partial_{n_E}\varphi_h - \alpha h_E^{-1}\varphi_h\big)_E \\
&- \sum_{E\in\mathcal{E}_h(\partial T\cap\Gamma_N)} \big(u^N - \hat{u}_E^N, \varphi_h\big)_E \\
&+ \alpha \sum_{E\in\mathcal{E}_h(T)} h_E^{-1} \ ([u_h - u_H], [\varphi_h])_E \\
&+ \alpha \sum_{E\in\mathcal{E}_h(\partial T\setminus\Gamma_D)} h_E^{-1} \ ([u_H], [\varphi_h])_E.
\end{aligned}
$$

In view of (4.4b), the inverse inequality and the trace inequalities imply that for $1 \leq i \leq 4$

$$
\|\nabla\varphi_h\|_{T_i}^2 \lesssim h_{T_i}^2 \ \left\|\hat{f}_T + \Delta u_H\right\|_{T_i}^2, \tag{4.9a}
$$

$$
\|[\varphi_h]\|_E^2 \lesssim h_E^3 \ \left\|\hat{f}_T + \Delta u_H\right\|_{T_i}^2, \ E \in \mathcal{E}_h(\partial T_i), \tag{4.9b}
$$

$$
\|\{\partial_{n_E}\varphi_h\}\|_E^2 \lesssim h_E \ \left\|\hat{f}_T + \Delta u_H\right\|_{T_i}^2, \ E \in \mathcal{E}_h(\partial T_i). \tag{4.9c}
$$

Then, using (4.4b) and (4.9a)–(4.9c), straightforward estimation of the terms on the right-hand side in (4.8) gives the assertion. ▯

LEMMA 4.4. *Let $E \in \mathcal{E}_H(\Omega), E = T_+ \cap E_-, T_\pm \in \mathcal{T}_H(\Omega)$, be a refined edge and $\omega_E := T_+ \cup T_-$. Then, there holds*

$$
\begin{aligned}
h_E\|[\partial_{n_E} u_H]\|_E^2 \ &\lesssim \sum_{T_\pm\in\mathcal{T}_H(\omega_E)} \eta_{T_\pm}^2 + \alpha \sum_{E'\in\mathcal{E}_H(\omega_E\cap\Omega)} \eta_{E',2}^2 \\
&+ \ \alpha \sum_{E\in\mathcal{E}_H(\partial\omega_E\cap\Gamma_D)} \eta_{E',D}^2 + \sum_{E\in\mathcal{E}_H(\partial\omega_E\cap\Gamma_N)} osc_{E'}^2(u^N).
\end{aligned} \tag{4.10}
$$

For a refined edge $E \in \mathcal{E}_H(\Gamma_N)$ with $E = \partial T \cap \Gamma_N, T \in \mathcal{T}_H(\Omega)$, we have

$$
h_E\|u^N - \partial_{n_E} u_H\|_E^2 \ \lesssim \ \eta_T^2 \ + \ \alpha \sum_{E'\in\mathcal{E}_H(T\cap\Omega)} \eta_{E',2}^2. \tag{4.11}
$$

Proof. For the proof of (4.10) let us assume that $E = T_+ \cap T_-, T_\pm \in \mathcal{T}_H(\Omega)$. We choose $\varphi_H \in CR_p(\Omega; \mathcal{T}_H(\Omega))$ as a linear combination of the basis functions associated with the edge E such that

$$h_E \|[\partial_{n_E} u_H]\|_E^2 = ([\partial_{n_E} u_H], \varphi_H)_E, \tag{4.12a}$$

$$\|\varphi_H\|_{T_\pm} \lesssim h_E^{3/2} \|[\partial_{n_E} u_H]\|_E, \tag{4.12b}$$

$$(q_{E'}, \varphi_H)_{E'} = 0, \quad q_{E'} \in \Pi_{p-1}(E') \tag{4.12c}$$

for any edge E'. Using the definition of $\partial_{n_E} u_H$, it follows that

$$\begin{aligned} &([\partial_{n_E} u_H], \varphi_H)_E \\ &= (\nu_E^+ \cdot \nabla u_H^+, \varphi_H)_E \ + \ (\nu_E^- \cdot \nabla u_H^-, \varphi_H)_E. \end{aligned} \tag{4.13}$$

By Green's formula, we find

$$\begin{aligned} &(\nabla u_H, \nabla \varphi_H)_{T_\pm} \\ &= \ -(\Delta u_H, \varphi_H)_{T_\pm} \ + \sum_{E' \in \mathcal{E}_H(\partial T_\pm)} (\partial_{n_{E'}} u_H, \varphi_H)_{E'}. \end{aligned} \tag{4.14}$$

By (4.12c) we have

$$(\partial_{n_{E'}} u_H, \varphi_H)_{E'} \ = \ 0, \quad E' \in \mathcal{E}_H(\partial \omega_E), \tag{4.15}$$

whence

$$\begin{aligned} &h_E \ \|[\partial_{n_E} u_H]\|_E^2 \\ &= \ ([\partial_{n_E} u_H], \varphi_H)_E \ = \ (\nabla u_H, \nabla \varphi_H)_{\omega_E} \ + \ (\Delta u_H, \varphi_H)_{\omega_E}. \end{aligned} \tag{4.16}$$

On the other hand, since φ_H is an admissible test function in (2.4), we have

$$\begin{aligned} (\nabla u_H, \nabla \varphi_H)_{\omega_E} = (f, \varphi_H)_{\omega_E} &+ \sum_{E' \in \mathcal{E}_H(\partial\omega_E \cap \Gamma_N)} \big(u^N, \varphi_H\big)_{E'} \\ - \sum_{E' \in \mathcal{E}_H(\partial\omega_E \cap \Gamma_D)} &\big(u^D, \partial_{n_{E'}} \varphi_H - \alpha h_{E'}^{-1} \varphi_H\big)_{E'} \\ + \sum_{E' \in \mathcal{E}_H(\omega_E)} ([u_H], \{\partial_{n_{E'}} \varphi_H\})_{E'} &- \alpha \sum_{E' \in \mathcal{E}_H(\partial\omega_E)} h_{E'}^{-1} ([u_H], [\varphi_H])_{E'}, \end{aligned} \tag{4.17}$$

where we have used (4.15) and (4.12c) on E. Combining (4.16) and (4.17) results in

$$\begin{aligned} h_E \ \|\{\partial_{n_E} u_H\}\|_E^2 \ &= \ (f + \Delta u_H, \varphi_H)_{\omega_E} \\ &+ \sum_{E' \in \mathcal{E}_H(\omega_E \setminus \Gamma_D)} ([u_H], \{\partial_{n_{E'}} \varphi_H\})_{E'} \\ &- \alpha \sum_{E' \in \mathcal{E}_H(\partial\omega_E \setminus \Gamma_D)} h_{E'}^{-1} ([u_H], [\varphi_H])_{E'} \\ &+ \sum_{E' \in \mathcal{E}_H(\partial\omega_E \cap \Gamma_N)} \big(u^N - \hat{u}_{E'}^N, \varphi_H\big)_{E'} \\ &- \sum_{E' \in \mathcal{E}_H(\partial\omega_E \cap \Gamma_D)} \big(u^D - u_H, \partial_{n_{E'}} \varphi_H - \alpha h_{E'}^{-1} \varphi_H\big)_{E'}. \end{aligned} \tag{4.18}$$

Observing (4.12b), the trace inequalities yield

$$\|[\varphi_H]\|_{E'} \lesssim h_E \|[\partial_{n_E} u_H]\|_E, \qquad E' \in \mathcal{E}_H(\partial\omega_E), \tag{4.19a}$$

$$\|\{\partial_{n_{E'}} \varphi_H\}\|_{E'} \lesssim \|[\partial_{n_E} u_H]\|_E, \qquad E' \in \mathcal{E}_H(\partial\omega_E). \tag{4.19b}$$

Taking advantage of (4.12b), (4.19a), and (4.19b), the assertion can be deduced by straightforward estimation of the terms on the right-hand side in (4.18). The proof of (4.11) follows by similar arguments. ∎

LEMMA 4.5. *Let $E \in \mathcal{E}_H(\Omega), E = T_+ \cap T_-, T_\pm \in \mathcal{T}_H(\Omega)$, be a refined edge and $\omega_E := T_+ \cup T_-$. Then, there holds*

$$\alpha h_E^{-1} \|[u_H]|_E\|_E^2 \lesssim a_h|_{\omega_E}(u_h - u_H, u_h - u_H) + \alpha \sum_{E' \in \mathcal{E}_h(E)} h_{E'}^{-1} \|[u_h]\|_{E'}^2. \tag{4.20}$$

Likewise, if $E \in \mathcal{E}_H(\Gamma_D)$ is a refined edge such that $E = \partial T \cap \Gamma_D, T \in \mathcal{T}_H(\Omega)$, there holds

$$\begin{aligned} \alpha h_E^{-1} \|u^D - u_H\|_E^2 \lesssim\ & a_h|_T(u_h - u_H, u_h - u_H) \\ & + \alpha \sum_{E' \in \mathcal{E}_h(E)} h_{E'}^{-1} \|u^D - u_h\|_{E'}^2 + \mathrm{osc}_E^2\left(u^D\right). \end{aligned} \tag{4.21}$$

Proof. For the proof of (4.20), choose $\psi_H^\pm \in CR_p(\Omega; \mathcal{T}_H(\Omega))$ with $\mathrm{supp}(\psi_H^\pm) = T_\pm$ as a linear combination of basis functions associated with E such that

$$\left([u_H], \psi_H^\pm\right)_E = \pm\tfrac{1}{2}\|[u_H]\|_E^2, \tag{4.22a}$$

$$\left\|\psi_H^\pm\right\|_{T_\pm} \lesssim h_E^{1/2}\|[u_H]\|_E. \tag{4.22b}$$

We define $\varphi_H \in V_H$ by $\varphi_H|_{T_\pm} = \psi_H^\pm$ and $\varphi_H|_T \equiv 0, T \in \mathcal{T}_H(\Omega) \setminus \{\omega_E\}$. Then, it follows from (4.22a) that

$$\alpha h_E^{-1} \|[u_H]|_E\|_E^2 \;=\; \alpha h_E^{-1}([u_H], [\varphi_H])_E. \tag{4.23}$$

Since φ_H is an admissible test function in (2.4), we have

$$\begin{aligned} & \alpha h_E^{-1}([u_H], [\varphi_H])_E \;= \\ & - (\nabla u_H, \nabla \varphi_H)_{\omega_E} \;+\; (f, \varphi_H)_{\omega_E} \\ & + \sum_{E' \in \mathcal{E}_H(\partial\omega_E \cup \{E\})} (\{\partial_{n_{E'}} u_H\}, [\varphi_H])_{E'} \\ & + \sum_{E' \in \mathcal{E}_H(\partial\omega_E) \cup \{E\}} ([u_H], \{\partial_{n_{E'}} \varphi_H\})_{E'} \\ & - \alpha \sum_{E' \in \mathcal{E}_H(\partial\omega_E)} h_{E'}^{-1}([u_H], [\varphi_H])_{E'} \\ & + \sum_{E' \in \mathcal{E}_H(\partial\omega_E \cap \Gamma_N)} \left(u^N, \varphi_H\right)_{E'} \\ & - \alpha \sum_{E' \in \mathcal{E}_H(\partial\omega_E \cap \Gamma_D)} \left(u^D, \partial_{n_{E'}} \varphi_H - \alpha h_{E'}^{-1} \varphi_H\right)_{E'}. \end{aligned} \tag{4.24}$$

On the other hand, $(\varphi_H|_{T'})_{T' \in \mathcal{T}_h(\Omega)}$ is an admissible test function in the fine grid equation (2.4). Hence, observing $[\varphi_H] = 0$ and $[u_H] = 0$ on $E' \in \mathcal{E}_h(\mathrm{int}(T_\pm))$, we get

$$
\begin{aligned}
0 = {} & (\nabla u_h, \nabla \varphi_H)_{\omega_E} \; - \; (f, \varphi_H)_{\omega_E} \\
& - \sum_{E' \in \mathcal{E}_h(\partial\omega_E \cup \{E\})} (\{\partial_{n_{E'}} u_h\}, [\varphi_H])_{E'} \\
& - \sum_{E' \in \mathcal{E}_h(\partial\omega_E \cup \{E\})} ([u_h], \{\partial_{n_{E'}} \varphi_H\})_{E'} \\
& - \sum_{E' \in \mathcal{E}_h(int(\omega_E) \setminus \{E\})} ([u_h - u_H], \{\partial_{n_{E'}} \varphi_H\})_{E'} \\
& + \alpha \sum_{E' \in \mathcal{E}_h(\partial\omega_E \cup \{E\})} h_{E'}^{-1} ([u_h], [\varphi_H])_{E'} \\
& - \sum_{E' \in \mathcal{E}_h(\partial\omega_E \cap \Gamma_N)} \left(u^N, \varphi_H\right)_{E'} \\
& + \alpha \sum_{E' \in \mathcal{E}_h(\partial\omega_E \cap \Gamma_D)} \left(u^D, \partial_{n_{E'}} \varphi_H - \alpha h_{E'}^{-1} \varphi_H\right)_{E'} .
\end{aligned}
\tag{4.25}
$$

In view of (4.22b), the inverse inequality and the trace inequalities imply

$$\left\|\nabla \psi_H^\pm\right\|_{T_\pm} \lesssim h_E^{-1/2} \|[u_H]\|_E, \tag{4.26a}$$

$$\left\|\psi_H^\pm\right\|_{E'} \lesssim \|[u_H]\|_E, \qquad E' \in \mathcal{E}_H(\omega_E), \tag{4.26b}$$

$$\left\|\partial_{n_{E'}} \psi_H^\pm\right\|_{E'} \lesssim h_E^{-1} \|[u_H]\|_E, \qquad E' \in \mathcal{E}_H(\omega_E). \tag{4.26c}$$

Combining (4.24) and (4.25) and using (4.22b) and (4.26a)–(4.26c), straightforward estimation gives the assertion. The proof of (4.21) can be established similarly. ◻

Remark 4.6. The proof of the perturbed discrete local efficiency is carried out under the assumption of geometrically conforming meshes. However, the fact that in Lemmas 4.4 and 4.5 the admissible test functions for the fine grid equation are chosen as linear combinations of the coarse grid Crouzeix–Raviart basis functions allows the handling of hanging nodes as well.

5. Proof of the error reduction property. In the convergence analysis of standard FEMs [18, 30], the proof of the error reduction property makes essential use of Galerkin orthogonality which in the framework of IPDG reads as follows:

$$a_h(u_h - u_H, u_h - u_H) \; = \; a_h(e_H, e_H) \; - \; a_h(e_h, e_h). \tag{5.1}$$

However, we measure the error e_H with respect to the mesh-dependent energy norm $a_H(\cdot,\cdot)$ associated with the coarse mesh $\mathcal{T}_H(\Omega)$, and, hence, (5.1) cannot be used directly. It is known from the convergence analysis of adaptive nonconforming finite elements [10] or of mixed finite elements [11] that in the absence of Galerkin orthogonality convergence can be established provided some sort of perturbed Galerkin orthogonality holds true. For the IPDG under consideration, we can rewrite (5.1) according to

$$a_h(u_h - u_H, u_h - u_H) \; = \; (1 + \delta_{h,H}(e_H)) \; a_H(e_H, e_H) \; - \; a_h(e_h, e_h), \tag{5.2}$$

where in the case $a_H(e_H, e_H) \neq 0$ the perturbation term $\delta_{h,H}(e_H)$ is given by

$$\delta_{h,H}(e_H) \;:=\; \frac{a_h(e_H, e_H) - a_H(e_H, e_H)}{a_H(e_H, e_H)}. \tag{5.3}$$

We would be able to conclude, if we can show that $\delta_{h,H}(e_H)$ can be made sufficiently small.

LEMMA 5.1 (perturbed Galerkin orthogonality). *There exists a positive constant C_1 depending only on the local geometry of the triangulations such that for the perturbation term $\delta_{h,H}(e_H)$ there holds*

$$\delta_{h,H}(e_H) \;\le\; \frac{C_1}{\alpha}. \tag{5.4}$$

Proof. Following the reasoning in [28, Proposition 4.1], we can easily show

$$\begin{aligned} a_h(e_H, e_H) \;&\le\; a_H(e_H, e_H) \\ &+ c_1\,\alpha \left(\sum_{E \in \mathcal{E}_H(\Omega)} h_E^{-1} \|[u_H]\|_E^2 + \sum_{E \in \mathcal{E}_H(\Gamma_D)} h_E^{-1} \|u^D - u_H\|_E^2 \right), \end{aligned}$$

where $c_1 > 0$ is a constant depending only on the local geometry of the triangulations. On the other hand, the local efficiency of the residual estimator (cf. [27]) tells us that there exists another positive constant c_2 which also depends only on the local geometry of the triangulations such that

$$\begin{aligned} &\sum_{E \in \mathcal{E}_H(\Omega)} h_E^{-1} \|[u_H]\|_E^2 \\ &+ \sum_{E \in \mathcal{E}_H(\Gamma_D)} h_E^{-1} \left\|u^D - u_H\right\|_E^2 \;\le\; \frac{c_2}{\alpha^2}\, a_H(e_H, e_H). \end{aligned} \tag{5.5}$$

Combining the two preceding estimates allows us to conclude with $C_1 := c_1 c_2$. ∎

Proof of Theorem 3.1. The reliability, the bulk criterion, and the discrete local efficiency infer the existence of a positive constant C_2 depending only on γ, Θ, and the local geometry of the triangulations such that

$$\begin{aligned} a_H(e_H, e_H) \;\le\; C_2 \Bigg(& a_h(u_h - u_H, u_h - u_H) + \operatorname{osc}_H^2 \\ &+ \alpha \sum_{E \in \mathcal{E}_h(\Omega)} h_E^{-1} \|[u_h]\|_E^2 \;+\; \alpha \sum_{E \in \mathcal{E}_h(\Gamma_D)} h_E^{-1} \|u_D - u_h\|_E^2 \Bigg). \end{aligned}$$

Using (5.2), (5.4), and (5.5) with h instead of H, we obtain the existence of a positive constant C_3 such that

$$a_H(e_H, e_H) \;\le\; C_2 \left(1 + \frac{2C_1}{\alpha}\right) a_H(e_H, e_H) - \left(C_2 - \frac{C_3}{\alpha}\right) a_h(e_h, e_h) + C_2 \operatorname{osc}_H^2,$$

from which we deduce

$$a_h(e_h, e_h) \le \left(C_2 - \frac{C_3}{\alpha}\right)^{-1} \left[\left(\left(1 + \frac{2C_1}{\alpha}\right) C_2 - 1\right) a_H(e_H, e_H) + C_2 \operatorname{osc}_H^2\right].$$

For $\alpha > 2C_1C_2 + C_3$, the error reduction property (3.10) results with $\rho_1 := (C_2 - \frac{C_3}{\alpha})^{-1}((1 + \frac{2C_1}{\alpha})C_2 - 1) < 1$. ▯

Remark 5.2. We note that $a_h(\cdot, \cdot)$ is not coercive on the energy space. However, it can be shown (cf. Proposition 4.2 in [28]) that $\|e_h\|^2_{1,h,\Omega} \lesssim a_h(e_h, e_h)$.

Remark 5.3. The coefficients $C_i, 1 \le i \le 3$, depend on the local geometry of the triangulations which is determined by the initial coarse triangulation and the refinement process. Moreover, C_2 depends on γ in (2.8) and on $0 < \Theta \le 1$ in (3.8a) and (3.8b). For an appropriate initial triangulation, we can expect $C_i = O(1), 1 \le i \le 3$, so that the requirement $\alpha > C_1C_2 + C_3$ results in values of α of approximately the same magnitude as required for the coercivity of the mesh-dependent bilinear forms. This is confirmed by the numerical results in section 6.

6. Computational results. In the following numerical experiments, we used the bisection algorithm, for all test cases, derived from the *AFEM@Matlab* implementation [14].

We verify the suitability of our theoretical results using standard test cases (see [10]). They are studies of the behavior of the algorithm in the case of the standard singularities induced by a reentrant corner of the domain. The right-hand side is chosen to be zero, and, therefore, data oscillations are present only on the boundary edges not adjacent to the singularity, where values of the analytical solutions are prescribed. The penalty parameter α has been chosen according to $\alpha = 15(p+1)^2$ as dictated by the coercivity requirement (2.8) (cf. also Remark 5.3).

First, we study the L-shaped domain with Dirichlet data $u^D = 0$ on the two edges adjacent to the reentrant corner and Neumann data on the remaining boundary. The refinement parameter is chosen as $\Theta = 0.6$. Table 6.1 shows a decline of the energy norm for this case by a factor of about 2/3 in each refinement step. This factor is only slightly better for quartic shape functions, confirming that the reduction rate depends mostly on Θ. Nevertheless, the meshes for P_4 are growing much slower, and in both cases we obtain the optimal approximation rates in terms of N_{dof}, namely, $N_{\text{dof}}^{-1/2}$ and N_{dof}^{-2}. Data oscillation occurs only at the outer boundary and is negligible in this case.

TABLE 6.1

Decline of the energy norm and data oscillation in terms of the refinement step, polynomial degrees 1 and 4.

	P_1			P_4		
l	N_{dof}	$\|e_l\|_A$	osc_l	N_{dof}	$\|e_l\|_A$	osc_l
0	36	2.81e − 1	9.32e − 2	180	6.07e − 2	9.32e − 2
1	114	1.98e − 1	6.83e − 2	570	3.83e − 2	6.83e − 2
2	252	1.39e − 1	3.35e − 2	960	2.60e − 2	5.38e − 2
3	630	9.53e − 2	1.66e − 2	1440	1.75e − 2	3.35e − 2
4	1428	6.48e − 2	1.07e − 2	2280	1.23e − 2	2.28e − 2
5	3180	4.42e − 2	5.85e − 3	3000	7.72e − 3	1.71e − 2
6	6714	3.00e − 2	4.10e − 3	4020	4.86e − 3	1.15e − 2
7	14076	2.08e − 2	2.46e − 3	5355	3.06e − 3	8.41e − 3
8	28368	1.43e − 2	1.51e − 3	6330	1.93e − 3	6.20e − 3
9	58461	9.91e − 3	9.18e − 4	7620	1.22e − 3	4.31e − 3

TABLE 6.2
Decline of the energy and data oscillation in terms of the refinement step, polynomial degrees 1 and 4.

	P_1			P_4		
l	N_{dof}	$\|e_l\|_A$	osc_l	N_{dof}	$\|e_l\|_A$	osc_l
0	18	8.83e-1	2.32e-1	90	5.16e-1	2.37e-1
1	48	6.90e-1	2.09e-1	165	4.39e-1	2.20e-1
2	138	5.76e-1	1.80e-1	285	3.61e-1	1.95e-1
3	219	4.86e-1	1.61e-1	690	3.03e-1	1.83e-1
...						
18	54804	4.42e-2	3.59e-2	32400	2.27e-2	4.69e-2
19	76809	3.73e-2	3.21e-2	39630	1.91e-2	4.22e-2
20	106821	3.16e-2	2.88e-2	52605	1.59e-2	3.79e-2
21	149829	2.67e-2	2.57e-2	63480	1.34e-2	3.41e-2

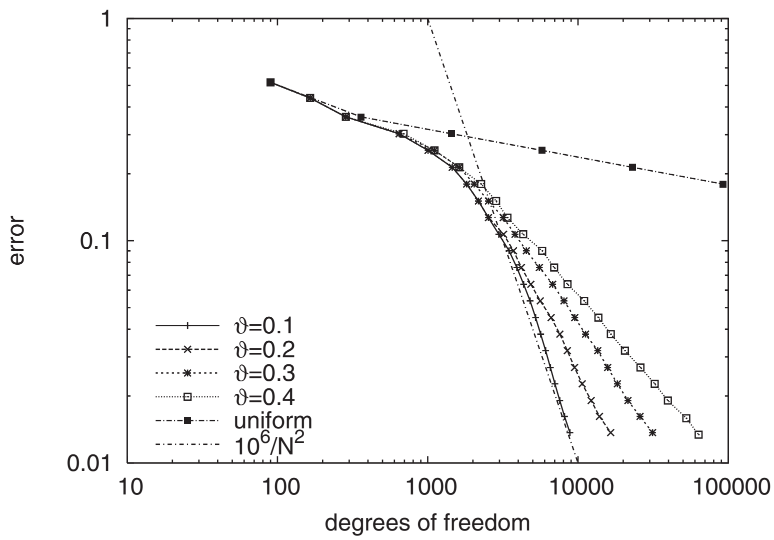

FIG. 6.1. *Error versus number of degrees of freedom for the slit domain, quartic polynomials.*

Next, we study the higher singularity of the slit domain. Here, Dirichlet boundary conditions are used on the whole boundary. Table 6.2 shows that for $\Theta = 0.4$ we obtain again constant error reduction rates.

The solution in this example is highly singular, and we expect that at least for higher order polynomials the refinement should be very local. Indeed, Figure 6.1 shows that the parameter Θ must be chosen carefully in order to obtain the optimal approximation with respect to the degrees of freedom, confirming results from [36] for standard AFEM. Only the very small value of $\Theta = 0.1$ is able to reproduce the optimal convergence order of N^{-2}. Figure 6.2 shows that this corresponds asymptotically to adding only 1/16 of the current number of cells in each step.

Even with the small size of $\Theta = 0.1$, the optimal N-term approximation rate is obtained only after several thousand degrees of freedom. Comparing Figures 6.1 and 6.2, we note that this corresponds to the fact that the bulk criterion refines much faster than the asymptotic rate in its initialization phase. On the other hand, this fast refinement allows the method to reach the asymptotic regime in only about 10 steps. Figure 6.3 shows that the refinement for $\Theta = 0.1$ is much more concentrated

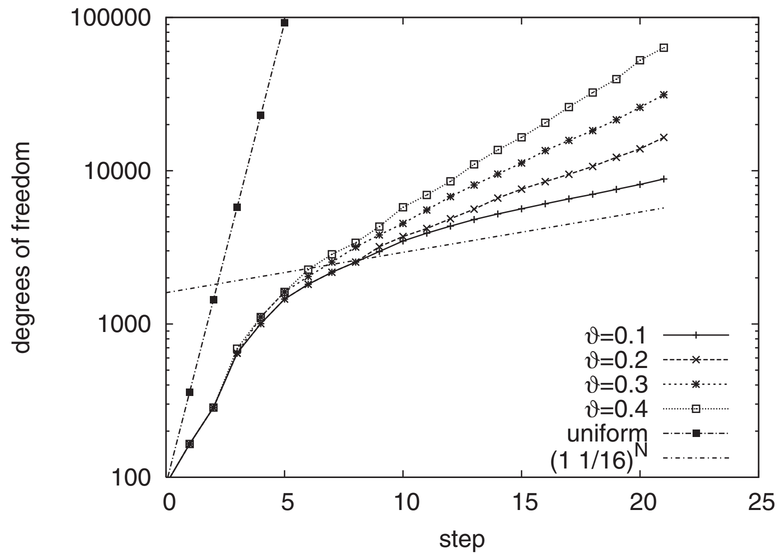

FIG. 6.2. *Development of mesh sizes during adaptive refinement, quartic polynomials.*

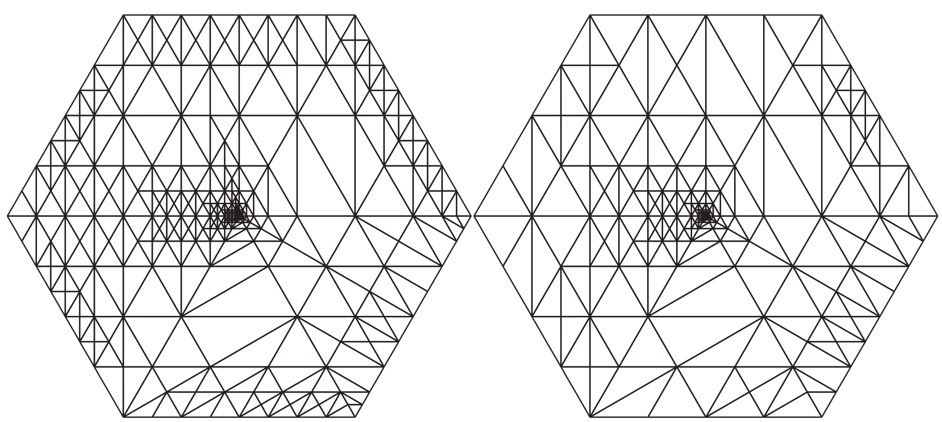

FIG. 6.3. *Meshes for the slit domain, polynomial degree* 4, $\Theta = 0.4$ *(left) and* $\Theta = 0.1$ *(right).*

at the central singularity, while $\Theta = 0.4$ puts more weight in reducing the boundary projection errors.

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