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DIVERGENCE-CONFORMING DISCONTINUOUS GALERKIN METHODS AND C^0 INTERIOR PENALTY METHODS*

GUIDO KANSCHAT† AND NATASHA SHARMA†

Abstract. In this paper, we show that recently developed divergence-conforming methods for the Stokes problem have discrete stream functions. These stream functions in turn solve a continuous interior penalty problem for biharmonic equations. The equivalence is established for the most common methods in two dimensions based on interior penalty terms. Then, extensions of the concept to discontinuous Galerkin methods defined through lifting operators, for different weak formulations of the Stokes problem, and to three dimensions are discussed. Application of the equivalence result yields an optimal error estimate for the Stokes velocity without involving the pressure. Conversely, combined with a recent multigrid method for Stokes flow, we obtain a simple and uniform preconditioner for harmonic problems with simply supported and clamped boundary.

Key words. finite element cochain complex, interior penalty methods, Stokes equations, biharmonic, divergence-free solutions, preconditioning, error estimates

AMS subject classifications. 65N30, 65N15, 65N55, 76D07, 74K20

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1. Introduction.

In this article, we show how finite element exterior calculus, when applied to interior penalty methods, provides a formalism to obtain algebraically exact stream functions for discrete incompressible flow problems. As our key theorem, we show that under the assumption of a simply connected domain, the solution to an H^{div}-conforming discontinuous Galerkin (H^{div}-DG) scheme for the Stokes equations is the curl of the solution to a corresponding $\mathcal{C}^0$ interior penalty ($\mathcal{C}^0$-IP) formulation of a corresponding biharmonic problem. The merit of this result rests not so much on its mathematical depth as on its application to unifying the analytical tools for both equations. Thus, in the main section of this article we exemplify the usefulness of the theorem at hand of several corollaries, where known results for one of the methods yield new information on the other.

For several years, H^{div}-DG methods for incompressible flow and $\mathcal{C}^0$-IP methods have been developed independently of each other. The latter were introduced by Engel et al. in [22] and refined and analyzed by Brenner and coworkers in [12, 13, 14, 15, 16, 17]. This series of publications contains the a priori and a posteriori analysis, as well as multigrid and domain decomposition solvers for the plate bending problem. Recently, convergence of adaptive methods for the $\mathcal{C}^0$-IP method has been studied in [26]. On the other hand, a priori and a posteriori error analysis for H^{div}-DG methods was provided in [19, 20, 31, 32, 35]. More recently, monolithic multigrid methods for these formulations have been studied in [33].

Here, we use the technique of finite element exterior calculus [6, 7] to study the relation between the two methods. Algebraic comparison of the involved bilinear forms

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†Interdisziplinäres Zentrum für Wissenschaftliches Rechnen (IWR), Ruprecht-Karls-Universität Heidelberg, 69120 Heidelberg, Germany (guido.kanschat@iwr.uni-heidelberg.de, natasha.sharma@iwr.uni-heidelberg.de). The first author was supported in part by the National Science Foundation through grant DMS-0810387 and by the King Abdullah University of Science and Technology (KAUST) through award KUS-C1-016-04. Part of this research was conceived and prepared when the author visited the Institute for Mathematics and Its Applications in Minneapolis.

reveals that the solutions ψ_h and $\mathbf{u}_h$ to the $\mathcal{C}^0$-IP and H^{div}-DG methods, respectively, are related by $\mathbf{u}_h = \nabla\times\psi_h$. The only assumption is that the domain be simply connected and two-dimensional. Even if this assumption does not hold, localization enables us to relate many of the results of one method to the other. By a similar exterior calculus argument, a divergence-free Stokes discretization was developed by Wang, Wang, and Ye in [44]; contrary to the method described there, we are not going to eliminate the pressure from our computations, but rather we develop theoretical tools allowing us to ignore it, for instance, in the subsection on error estimates. Therefore, our results remain meaningful even for three-dimensional Stokes problems, where computing the stream function would require solving a mixed problem as well.

The results in this article are intricately related to the work by Falk and Morley [23], in which similar equivalences are shown for nonconforming methods, namely, the Crouzeix–Raviart and the Morley elements. Here as there, a cochain complex of finite element spaces with associated bilinear forms can be exploited to exhibit relations between methods for the biharmonic and the Stokes problems, respectively. Without following this track, we point out that this suggests that stabilization techniques, which were suggested by Hansbo and Larson in [30] for the Crouzeix–Raviart element could be used for the Morley element when applied to the symmetric gradient of the curl operator.

Related are also methods which build a cochain complex with an H^1-conforming velocity space. Here, we would like to point out the sequence consisting of the Hsieh–Clough–Tocher and the Scott–Vogelius elements [42, 43] on simplices (see Zhang [45]) and the element discussed by Austin, Manteuffel, and McCormick in [9] using Bogner–Fox–Schmidt elements and a higher continuity version of the Raviart–Thomas element on rectangular meshes. The element by Zhang [46] is a very close cousin as well. In all these methods, the cochain complex guarantees that the key relation $\nabla\cdot\mathbf{V}_h = Q_h$ remains valid. Most recently, Stokes complexes were introduced by Falk and Neilan in [24] and Guzmán and Neilan in [29] to study the relationship between the divergence-free Stokes problem and the biharmonic problem. The works cited in this paragraph have in common that a cochain complex starting with an H^2-conforming finite element is used to obtain divergence-free solutions to the Stokes problem in H^1. This article on the other hand explores complexes based only on H^1-conforming stream functions, augmented by penalty terms for consistency.

This article is organized as follows. In section 2, we introduce the basic notation and review the stream function formulation of the two-dimensional Stokes problem. The main results on the equivalence of the two methods, Theorem 3.2 and Corollary 3.3, are derived in section 3. The main body of the article is section 4, where we discuss consequences of these results for error estimation and multigrid methods as well as their extension to alternative formulations and three dimensions.

2. Notation and stream function formulations.

Let Ω be a two-dimensional, polygonal domain with boundary $\partial\Omega = \Gamma_F \cup \Gamma_S \cup \Gamma_N$. In the following, $H^s(\Omega)$ denotes the L^2-based Sobolev space of differentiation order $s \geq 0$. If necessary, we will denote vector and tensor valued Sobolev spaces by $H^s(\Omega,\mathbb{R}^2)$ and $H^s(\Omega,\mathbb{R}^{2\times 2})$. The L^2-inner product is denoted by

$$(f,g) = (f,g)_{L^2(\Omega)} := \int_\Omega f \odot g \, d\mathbf{x}, \tag{2.1}$$

where the generic multiplication operator $\odot$ denotes the product, the dot product, or the double contraction for scalar, vector, and tensor functions, respectively. Inner products in other spaces are denoted by an index. In particular, on subspaces of $H^1(\Omega)$ with boundary conditions such that Friedrichs' inequality can be applied, we use the inner product

$$(f,g)_{H_0^1} = (\nabla f, \nabla g)\,.$$

We use standard differential operator notation for the gradient ∇p, the symmetric tensor of second derivatives $\nabla^2 p$, the divergence $\nabla\cdot\mathbf{v}$, and the Laplacian $\Delta p = \nabla\cdot\nabla p$. For vectors and tensors, we define

$$\nabla\mathbf{v} = \begin{pmatrix} \partial_1 v_1 & \partial_2 v_1 \\ \partial_1 v_2 & \partial_2 v_2 \end{pmatrix} \qquad \text{and} \qquad \nabla\cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \partial_1 a_{11} + \partial_2 a_{12} \\ \partial_1 a_{21} + \partial_2 a_{22} \end{pmatrix}.$$

The symmetric gradient of a vector function is $\mathbf{D}\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$. Additionally, we define the curl of a scalar function $\nabla\times p = (-\partial_2 p, \partial_1 p)^T$ and the rotation of a vector function $\nabla\times\mathbf{v} = \partial_2 v_1 - \partial_1 v_2$. We note that $\nabla\times\nabla\times p = -\Delta p$.

Functions $\mathbf{v} \in H^1(\Omega, \mathbb{R}^2)$ have boundary traces, and several essential boundary conditions are modeled by choosing subspaces with zero traces. We distinguish between the free boundary condition on Γ_F, the slip boundary condition $\mathbf{v}\cdot\mathbf{n} = 0$ on Γ_S, and the no-slip condition $\mathbf{v} = 0$ on Γ_N. Accordingly, we define the velocity space

$$\mathbf{V} = \left\{ \mathbf{v} \in H^1(\Omega) \;\middle|\; \mathbf{v}_{|\Gamma_S}\cdot\mathbf{n} = 0 \;\wedge\; \mathbf{v}_{|\Gamma_N} = 0 \right\}. \tag{2.2}$$

We explicitly allow that each of the components Γ_F, Γ_S, and Γ_N may be empty. Then, we define the pressure space

$$Q = \begin{cases} L^2(\Omega) & \text{if } \Gamma_F \neq \emptyset, \\ L^2(\Omega)/\mathbb{R} & \text{if } \Gamma_F = \emptyset. \end{cases}$$

2.1. The Stokes problem and its stream function formulations. We consider the Stokes problem in the following weak form: find $(\mathbf{u}, p) \in \mathbf{V}\times Q$ such that $\forall v \in V$ and $q \in Q$ there holds

$$2\left(\mathbf{D}\mathbf{u}, \mathbf{D}\mathbf{v}\right) - (p, \nabla\cdot\mathbf{v}) - (q, \nabla\cdot\mathbf{u}) = (\mathbf{f}, \mathbf{v})\,. \tag{2.3}$$

In the case that $\Gamma_N = \partial\Omega$, (2.3) can be reduced by integration by parts to

$$(\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) - (q, \nabla\cdot\mathbf{u}) = (\mathbf{f}, \mathbf{v})\,. \tag{2.4}$$

In order to characterize the divergence-free subspace of $\mathbf{V}$ by a Hodge decomposition, we first note that the sequence

$$\mathbb{R} \xrightarrow{\subset} H^2(\Omega) \xrightarrow{\nabla\times} H^1(\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega) \longrightarrow 0 \tag{2.5}$$

forms a Hilbert cochain complex; see, e.g., [24]. This complex is exact if Ω is simply connected. Adding appropriate boundary conditions, we introduce the space of stream functions

$$\Psi = \left\{ \psi \in H^2(\Omega) \;\middle|\; \psi_{|\Gamma_S\cup\Gamma_N} = 0 \;\wedge\; \partial_n\psi_{|\Gamma_N} = 0 \right\}. \tag{2.6}$$

For the subspaces with boundary conditions, we have the closed cochain complex

$$\Psi \xrightarrow{\nabla\times} \mathbf{V} \xrightarrow{\nabla\cdot} Q. \tag{2.7}$$

TABLE 1
Nomenclature and correspondence of boundary conditions for the biharmonic and the Stokes problem, respectively.

	Stokes		Biharmonic	
Γ_N	no-slip	$\mathbf{u} = 0$	clamped	$\varphi = \partial_n \varphi = 0$
Γ_S	slip	$\mathbf{u} \cdot \mathbf{n} = 0$	simply supported	$\varphi = \Delta\varphi = 0$
Γ_F	outflow	$\partial_n \mathbf{u} - p\mathbf{n} = 0$	free	$\Delta\varphi = \partial_n \Delta\varphi = 0$

See Table 1 for a correspondence and nomenclature of the boundary conditions. This sequence implies the Hodge decomposition

$$\mathbf{V} = \nabla^* Q \;\oplus\; \nabla\times\Psi \;\oplus\; \mathcal{H}, \tag{2.8}$$

where ∇^* is the adjoint of the divergence operator defined by

$$\forall \mathbf{v} \in \mathbf{V}\; \forall q \in Q: \;(\mathbf{v}, \nabla^* q)_{H^1_0} = (\nabla\cdot\mathbf{v}, q)_{H^1_0},$$

and $\mathcal{H} := \{\mathbf{w} \in \mathbf{V} \,\big|\, \nabla\cdot\mathbf{w} = 0 \text{ and } (\mathbf{w}, \nabla\times\psi) = 0\; \forall\psi \in \Psi\}$ is the space of harmonic forms in $\mathbf{V}$. Its dimension is equal to the number of holes in the domain. More details can be found in [7]. We refer to Figure 2, later in the paper, for a geometric idea of such functions. Moreover, $\mathcal{H} = \{0\}$ if Ω is simply connected. Below, we are going to investigate solutions to (2.3) and (2.4) restricted to the kernel of the divergence operator, for which we introduce the notation

$$\mathbf{V}^0 = \big\{\mathbf{v} \in \mathbf{V} \,\big|\, \nabla\cdot\mathbf{v} = 0\big\} = \nabla\times\Psi \;\oplus\; \mathcal{H}. \tag{2.9}$$

Remark 2.1. In order to simplify notation, and to avoid repeating references to the simply connected case, we will assume a simply connected domain for the remainder of this article, unless stated otherwise. Thus, from now on, $\mathcal{H} = \{0\}$ or $\mathbf{V}^0 = \nabla\times\Psi$. Since our arguments below are local, the existence of harmonic forms does not affect them in principle. Nevertheless, we point out that harmonic forms may pose a problem when the equivalence result in Theorem 3.2 is applied.

We eliminate the pressure by restricting the variational problems to the subspace $\mathbf{V}^0$. Using (2.9) and the previous remark, we replace trial functions by $\mathbf{u} = \nabla\times\varphi$ with $\varphi \in \Psi$ and test with functions $\mathbf{v} = \nabla\times\psi$. Then, (2.3) becomes

$$2\,(\mathbf{D}\nabla\times\varphi, \mathbf{D}\nabla\times\psi) = (\mathbf{f}, \nabla\times\psi)\,, \tag{2.10}$$

or, after applying $\mathbf{D}\nabla\times\varphi = \begin{pmatrix} -\partial_1\partial_2\varphi & -\partial_2^2\varphi \\ \partial_1^2\varphi & \partial_1\partial_2\varphi \end{pmatrix}$,

$$2\,((\partial_{11} - \partial_{22})\varphi, (\partial_{11} - \partial_{22})\psi) + 4\,(\partial_1\partial_2\varphi, \partial_1\partial_2\psi) = (\mathbf{f}, \nabla\times\psi)\,.$$

Simpler is the transformation of (2.4), which becomes

$$(\nabla\nabla\times\varphi, \nabla\nabla\times\psi) \equiv \big(\nabla^2\varphi, \nabla^2\psi\big) = (\mathbf{f}, \nabla\times\psi)\,. \tag{2.11}$$

Since for any $\varphi \in \Psi$ there holds $\nabla\varphi = 0$ on Γ_N, we can integrate the middle term by parts to obtain the alternative weak form

$$(\Delta\varphi, \Delta\psi) = (\mathbf{f}, \nabla\times\psi)\,. \tag{2.12}$$

A consequence of the stream function formulation is that the divergence-free velocity $\mathbf{u} = \nabla\times\varphi$ only depends on the divergence free part of $\mathbf{f}$ and can be computed ignoring the component of $\mathbf{f}$ in $\nabla^* Q$. The goal of the next section is to establish the same result for the discrete solution. But first, we have to introduce some notation.

2.2. Meshes, trace operators, and discrete spaces. In order to define finite element spaces, we first cover the domain Ω by a subdivision $\mathbb{T}_h$ into a mesh of triangular or quadrilateral cells T. For simplicity, we restrict ourselves to topologically conforming meshes such that each face F of a cell T is either a boundary face or a face of a second mesh cell. Extension to meshes with so-called hanging nodes is straightforward and has been widely discussed in the literature. The set of faces of $\mathbb{T}_h$ is denoted by $\mathbb{F}_h$ and consists of interior faces $\mathbb{F}_h^i$ and boundary faces $\mathbb{F}_h^\partial$.

Given the mesh $\mathbb{T}_h$, we generalize the notions of continuous and differentiable function spaces to so-called broken spaces such that, for instance, $\mathcal{C}(\mathbb{T}_h)$ and $H^s(\mathbb{T}_h)$ are the spaces of functions such that the restriction to each mesh cell $T \in \mathbb{T}_h$ is in $\mathcal{C}(T)$ and $H^s(T)$, respectively. No additional continuity requirements are imposed between mesh cells. Furthermore, for any integer $k > 0$, we define, $\mathcal{P}_k(T), \mathcal{P}_k(T)^d, \mathcal{P}_k(T)^{d\times d}$ as the space of scalar, vector, and tensor valued polynomials on T of degree at most k.

Let F be a face in $\mathbb{F}_h^i$ such that the two cells T_1 and T_2 are adjacent to F in the point x. For a function $u \in \mathcal{C}(\mathbb{T}_h)$, let $u_1(x)$ and $u_2(x)$ be the traces of u in x from cells T_1 and T_2, respectively. Then, we define the sum operator $[\![u]\!](x) := u_1 + u_2$. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward normal vector to T_1 and T_2, respectively. Then, by nature of its definition, the sum operator applied to multiples of the normal vector transforms to a jump,

$$[\![u\mathbf{n}]\!] = u_1\mathbf{n}_1 + u_2\mathbf{n}_2 = (u_1 - u_2)\mathbf{n}_1, \qquad [\![\partial_n u]\!] = \partial_{n_1}(u_1 - u_2),$$

but $[\![\partial_n^2 u]\!] = \partial_{n_1}^2(u_1 + u_2) = \partial_{n_2}^2(u_1 + u_2)$.

For any normal vector $\mathbf{n} = (n_x, n_y)^T$ in two dimensions, we define a tangential vector $\mathbf{t} = (-n_y, n_x)$ and denote that as $\mathbf{n}_2 = -\mathbf{n}_1$, accordingly $\mathbf{t}_2 = -\mathbf{t}_1$.

On the mesh $\mathbb{T}_h$, we introduce the spaces Ψ_h, $\mathbf{V}_h$, and Q_h of continuous, divergence-conforming and discontinuous finite element functions, respectively. To this corresponds the discrete Hilbert cochain complex

$$\begin{array}{ccccc} \Psi_h & \xrightarrow{\nabla\times} & \mathbf{V}_h & \xrightarrow{\nabla\cdot} & Q_h \\ \subset\big\downarrow & & \subset\big\downarrow & & \subset\big\downarrow \\ H^1(\Omega) & \xrightarrow{\nabla\times} & H^{\mathrm{div}}(\Omega) & \xrightarrow{\nabla\cdot} & L^2(\Omega) \end{array} . \tag{2.13}$$

At the boundary, the discrete spaces are constrained, such that $\forall \psi \in \Psi_h$ and $\mathbf{v} \in \mathbf{V}_h$ holds $\psi = 0$ and $\mathbf{v}\cdot\mathbf{n} = 0$, respectively, on $\Gamma_S \cup \Gamma_N$. In the case that $\Gamma_F = \emptyset$, Q_h is restricted to the subspace of mean-value free functions.

Examples are the well-known Raviart–Thomas [38] and Brezzi–Douglas–Marini [18] families. Note that the discrete spaces in this sequence are embedded in Sobolev spaces of insufficient regularity to support (2.3) to (2.11). In order to cure this problem, interior penalty methods have been introduced to restore consistency. These will be reviewed in the following section. But first, we denote an important consequence of the discrete Hilbert cochain complex (2.13).

PROPOSITION 2.2 (discrete Hodge decomposition). *The space* $\mathbf{V}_h$ *admits an* L^2*-orthogonal decomposition into*

$$\mathbf{V}_h^0 = \left\{ \mathbf{v}_h \in \mathbf{V}_h \;\middle|\; \nabla\cdot\mathbf{v}_h = 0 \right\}$$

and its orthogonal complement $\mathbf{V}_h^\perp$*. The space* $\mathbf{V}_h^0$ *can be characterized as*

$$\mathbf{V}_h^0 = \nabla\times\Psi_h \;\oplus\; \mathcal{H}_h, \tag{2.14}$$

where $\mathcal{H}_h := \{\mathbf{w} \in \mathbf{V}_h \mid \nabla\cdot\mathbf{w} = 0$ *and* $(\mathbf{w}, \nabla\times\psi) = 0\ \forall\psi \in \Psi_h\}$ *are the discrete harmonic forms in* $\mathbf{V}_h$ *with* $\dim\mathcal{H}_h = \dim\mathcal{H}$. *The space* $\mathbf{V}_h^\perp$ *is characterized as*

$$\mathbf{V}_h^\perp = \big\{\mathbf{v}_h \in \mathbf{V}_h \mid \exists q_h \in Q_h\ \forall\mathbf{w}_h \in \mathbf{V}_h : (\mathbf{v}_h, \mathbf{w}_h) = (q_h, \nabla\cdot\mathbf{w}_h)\big\}. \tag{2.15}$$

Given a function $q_h \in Q_h$, we denote the function $\mathbf{v}_h$ defined by the weak form in equation (2.15) as $\nabla_h^* q_h$. In view of Remark 2.1, we assume $\mathcal{H}_h = \{0\}$ such that $\mathbf{V}_h^0 = \nabla\times\Psi_h$.

We close this section by introducing shorthand notation for integrals over meshes and sets of faces, namely,

$$(f, g)_{\mathbb{T}_h} := \sum_{T\in\mathbb{T}_h} (f, g)_T\,, \qquad \langle f, g\rangle_{\mathbb{F}_h} := \sum_{F\in\mathbb{F}_h} \langle f, g\rangle_F = \sum_{F\in\mathbb{F}_h} \int_F f \odot g\, ds,$$

where as in (2.1) the symbol $\odot$ is used as generic multiplication operator. Similarly, we introduce the seminorms

$$\|f\|_{\mathbb{T}_h} := \sqrt{(f, f)_{\mathbb{T}_h}} \qquad \text{and} \qquad \|f\|_{\mathbb{F}_h} := \sqrt{\langle f, f\rangle_{\mathbb{F}_h}}.$$

The face integrals and norms may be restricted to subsets $\mathbb{F}_h^i$ and $\mathbb{F}_h^\partial$ with the obvious definition.

3. From H^{div}-IP to $\mathcal{C}^0$-IP. First, we consider the H^{div}-conforming interior penalty method in [20, 34, 27] for the Stokes problem (2.4). Here, we consider solution pairs $(\mathbf{u}, p)$ in $\mathbf{V}_h \times Q_h$. In order to define the interior penalty method, we use the consistency and penalty terms

$$a_c^i(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\left\langle [\![\nabla\mathbf{u}]\!], [\![\mathbf{v}\otimes\mathbf{n}]\!]\right\rangle_{\mathbb{F}_h^i}, \quad a_p^i(\mathbf{u}, \mathbf{v}) = \left\langle \gamma_h^2\, [\![\mathbf{u}\otimes\mathbf{n}]\!], [\![\mathbf{v}\otimes\mathbf{n}]\!]\right\rangle_{\mathbb{F}_h^i}, \tag{3.1}$$

where the first term corresponds to the average of the normal derivative of $\mathbf{u}$, tested by the jump of $\mathbf{v}$. On the boundary $\partial\Omega = \Gamma_N$, we use the Nitsche terms

$$a_c^\partial(\mathbf{u}, \mathbf{v}) = \langle\nabla\mathbf{u}, \mathbf{v}\otimes\mathbf{n}\rangle_{\mathbb{F}_h^\partial}, \quad a_p^\partial(\mathbf{u}, \mathbf{v}) = 2\left\langle\gamma_h^2\mathbf{u}\otimes\mathbf{n}, \mathbf{v}\otimes\mathbf{n}\right\rangle_{\mathbb{F}_h^\partial}, \tag{3.2}$$

where γ_h is of the form

$$\gamma_h^2 = \frac{\gamma_0^2}{h_F}$$

with h_F a suitably defined mesh size on the face F, for instance, the minimum size of an adjacent mesh cell measured orthogonally to the face. The penalty parameter γ_0 is chosen sufficiently large to guarantee the stability of the DG formulation; see, for instance, [3]. Furthermore, for the purpose of error estimation, we will assume that γ_0 is chosen such that the estimate [36, Theorem 3.2(iv)] by Karakashian and Pascal holds.

The bilinear form for the Laplace operator is defined by

$$a_h^i(\mathbf{u}, \mathbf{v}) = a_p^i(\mathbf{u}, \mathbf{v}) - a_c^i(\mathbf{u}, \mathbf{v}) - a_c^i(\mathbf{v}, \mathbf{u}), \tag{3.3}$$

$$a_h^\partial(\mathbf{u}, \mathbf{v}) = a_p^\partial(\mathbf{u}, \mathbf{v}) - a_c^\partial(\mathbf{u}, \mathbf{v}) - a_c^\partial(\mathbf{v}, \mathbf{u}), \tag{3.4}$$

$$a_h(\mathbf{u}, \mathbf{v}) = (\nabla\mathbf{u}, \nabla\mathbf{v})_{\mathbb{T}_h} + a_h^i(\mathbf{u}, \mathbf{v}) + a_h^\partial(\mathbf{u}, \mathbf{v}), \tag{3.5}$$

and the H^{div}-IP formulation of the Stokes problem becomes the following: find $(\mathbf{u}, p) \in \mathbf{V}_h \times Q_h$ such that there holds

$$a_h(\mathbf{u}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) - (q, \nabla\cdot\mathbf{u}) = (f, \mathbf{v})_\Omega \quad \forall\, (\mathbf{v}, q) \in \mathbf{V}_h \times Q_h. \tag{3.6}$$

Since $\mathbf{V}_h$ is a subspace of a completely discontinuous polynomial space, we extract from [19, 20] the following.

PROPOSITION 3.1. *The bilinear form $a_h(.,.)$ is bounded and elliptic uniformly in h on $\mathbf{V}_h$ equipped with the norm $\|.\|_{1,h}$ defined by*

$$\|\mathbf{u}\|_{1,h}^2 = \|\nabla\mathbf{u}\|_{\mathbb{T}_h}^2 + a_p^i(\mathbf{u}, \mathbf{u}) + a_p^\partial(\mathbf{u}, \mathbf{u}). \tag{3.7}$$

The weak formulation (3.6) *has a unique discrete solution, which admits the stability estimate*

$$\|\mathbf{u}\|_{1,h} + \|p\| \le c\|f\|.$$

In order to exhibit the relation to fourth order problems, we exploit the sequence (2.13) and let $\mathbf{u} = \nabla\times\varphi$ and test with functions $\mathbf{v} = \nabla\times\psi$. The Stokes equation (3.6) then reduces to finding $\varphi \in \Psi_h$ such that $\forall\psi \in \Psi_h$ there holds

$$a_h(\nabla\times\varphi, \nabla\times\psi) = (f, \nabla\times\psi)_\Omega\,.$$

In order to analyze this form, we note that for functions in $\mathbf{v} \in \mathbf{V}_h$ there holds $[\![\mathbf{v}\cdot\mathbf{n}]\!] = 0$ on any interior edge $F \in F_h^i$ and $\mathbf{v}\cdot\mathbf{n} = 0$ on the boundary $\partial\Omega$. Thus, the interior penalty terms in (3.1) and (3.2) affect the tangential component only. Using

$$u_t = \mathbf{u}\cdot\mathbf{t} = \begin{pmatrix} -\partial_2\varphi \\ \partial_1\varphi \end{pmatrix} \cdot \begin{pmatrix} -n_y \\ n_x \end{pmatrix} = n_x\partial_1\varphi + n_y\partial_2\varphi = \partial_n\varphi,$$

and in local coordinates

$$\nabla\mathbf{u} : (\mathbf{v}\otimes\mathbf{n}) = \begin{pmatrix} \partial_n u_n & \partial_t u_n \\ \partial_n u_t & \partial_t u_t \end{pmatrix} : \begin{pmatrix} v_n & 0 \\ v_t & 0 \end{pmatrix} = \partial_n u_n\, v_n + \partial_n u_t\, v_t,$$

we obtain the edge terms

$$\begin{aligned}
b_c^\partial(\varphi, \psi) &= a_c^\partial(\nabla\times\varphi, \nabla\times\psi) = \frac{1}{2}\left\langle \partial_n^2\varphi, \partial_n\psi\right\rangle_{\mathbb{F}_h^\partial},\\
b_p^\partial(\varphi, \psi) &= a_p^\partial(\nabla\times\varphi, \nabla\times\psi) = \left\langle \gamma_h^2\partial_n\varphi, \partial_n\psi\right\rangle_{\mathbb{F}_h^\partial},\\
b_c^i(\varphi, \psi) &= a_c^i(\nabla\times\varphi, \nabla\times\psi) = \frac{1}{2}\left\langle \left[\!\left[\partial_n^2\varphi\right]\!\right], [\![\partial_n\psi]\!]\right\rangle_{\mathbb{F}_h^i},\\
b_p^i(\varphi, \psi) &= a_p^i(\nabla\times\varphi, \nabla\times\psi) = \left\langle \gamma_h^2\,[\![\partial_n\varphi]\!], [\![\partial_n\psi]\!]\right\rangle_{\mathbb{F}_h^i}.
\end{aligned} \tag{3.8}$$

Furthermore, the cell term yields

$$(\nabla\nabla\times\varphi, \nabla\nabla\times\psi)_{\mathbb{T}_h} = \left(\nabla^2\varphi, \nabla^2\psi\right)_{\mathbb{T}_h}$$

such that we conclude

$$\begin{aligned}
b_h(\varphi, \psi) &\equiv a_h(\nabla\times\varphi, \nabla\times\psi)\\
&= \left(\nabla^2\varphi, \nabla^2\psi\right)_{\mathbb{T}_h} - b_c^i(\varphi, \psi) + b_p^i(\varphi, \psi) - b_c^\partial(\varphi, \psi) + b_p^\partial(\varphi, \psi).
\end{aligned} \tag{3.9}$$

This is the bilinear form of the $\mathcal{C}^0$-IP method for biharmonic problems [14] with the mesh dependent norm $\|.\|_{2,h}$ given by

$$\|\varphi\|_{2,h}^2 = \|\nabla^2\varphi\|_{\mathbb{T}_h}^2 + b_p^i(\varphi,\varphi) + b_p^\partial(\varphi,\varphi). \tag{3.10}$$

Thus, we have proved the following.

THEOREM 3.2. *Let the spaces Ψ_h, $\mathbf{V}_h$, and Q_h on a two-dimensional mesh on a simply connected domain form a Hilbert cochain complex as in* (2.13) *with boundary conditions prescribed such that $\Gamma_N = \partial\Omega$. Then, if $\varphi \in \Psi_h$ is a solution to*

$$b_h(\varphi,\psi) = (\mathbf{f}, \nabla\times\psi) \quad \forall\psi\in\Psi_h \tag{3.11}$$

with $b_h(.,.)$ defined in (3.9)*, then $\mathbf{u} = \nabla\times\varphi \in \mathbf{V}_h$ is the unique velocity solution to the discrete Stokes problem* (3.6).

An immediate consequence of this theorem is the following corollary.

COROLLARY 3.3. *The velocity solution $\mathbf{u}_h \in \mathbf{V}_h$ and the pressure solution $p_h \in Q_h$ of the Stokes equation* (3.6) *can be computed independently. They solve the weak forms*

$$a_h(\mathbf{u}_h, \mathbf{v}_h^0) = \left(\mathbf{f}^0, \mathbf{v}_h^0\right) \qquad \forall \mathbf{v}_h^0 \in \mathbf{V}_h^0, \tag{3.12}$$

$$\left(\nabla p_h, \nabla_{a_h}^* q_h\right)_{\mathbb{T}_h} - \left\langle [\![p_h\mathbf{n}]\!], \nabla_{a_h}^* q_h\right\rangle_{\mathbb{F}_h^i} = \left(\mathbf{f}, \nabla_{a_h}^* q_h\right) \qquad \forall q_h \in Q_h, \tag{3.13}$$

respectively.

Proof. Obtaining (3.12) follows readily from (3.11) and by decomposing $\mathbf{f}$ into $\mathbf{f}^0 \in \mathbf{V}_h^0$ and $\mathbf{f}^\perp$ in its L^2-orthogonal complement. In order to compute the pressure, we define the $a_h(.,.)$-adjoint $\nabla_{a_h}^*$ of the divergence operator applied to a function $q \in Q_h$ by

$$a_h(\mathbf{v}, \nabla_{a_h}^* q) = (\nabla\cdot\mathbf{v}, q) \quad \forall \mathbf{v}\in\mathbf{V}_h. \tag{3.14}$$

Testing (3.6) with $\mathbf{v} = \nabla_{a_h}^* q$, using the fact that $\nabla\cdot\mathbf{u} = 0$, and using (3.14) yields

$$-\left(p, \nabla\cdot\nabla_{a_h}^* q\right) = \left(\mathbf{f}, \nabla_{a_h}^* q\right).$$

After integration by parts, we obtain the weak formulation (3.13). ☐

While the solution to the reduced problem (3.12) can be easily computed by either solving the biharmonic problem (3.11) or following [44] and constructing a basis of divergence-free velocities, we consider the pressure Poisson problem (3.13) more academic in nature. Indeed, while it is intuitively clear that the left-hand side of this formulation should be a discretization of the Laplacian, it is far less clear what this formulation actually looks like, or what its stability properties are. We emphasize that the above is more of a theoretical tool to obtain results on the discretizations and solvers rather than an approach compute the pressure or even the velocity.

We verify our theoretical results with numerical computations. To this end, we first set up a family of right-hand sides and solutions in the following way: let $p(t)$ be a polynomial to be chosen later. Then, the functions

$$\varphi(x,y) = p(x)p(y), \qquad \mathbf{u}(x,y) = \nabla\times\varphi(x,y) = \begin{pmatrix} p(x)p'(y) \\ -p'(x)p(y)\end{pmatrix}$$

solve (2.11) and (2.4), respectively, with right-hand side

$$\mathbf{f} = \mathbf{f}^0 = -\nabla\times\left(p''(x)p(y) + p(x)p''(y)\right). \tag{3.15a}$$

TABLE 2

The energy error $E_h = \|\nabla \mathbf{u} - \nabla \mathbf{u}_h\| = \|\nabla^2\varphi - \nabla^2\varphi_h\|$, the L^2-error $e_h = \|u - u_h\| = \|\nabla\varphi - \nabla\varphi_h\|$, and the difference $d_h = \|\nabla \times \varphi - \mathbf{u}\|$ for the problem with clamped boundary and different discretization orders on a sequence of mesh refinement levels L.

	Q_2 and $RT_1 \times Q_1$			Q_3 and $RT_2 \times Q_2$		
L	E_h	e_h	d_h	E_h	e_h	d_h
2	6.9e-0	1.2e-0	1.5e-11	1.7e-0	1.8e-1	8.3e-11
3	3.6e-0	3.8e-1	9.8e-11	4.2e-1	2.2e-2	2.6e-10
4	1.8e-0	1.1e-1	5.2e-10	1.0e-1	2.8e-3	8.2e-10
5	9.0e-1	3.0e-2	2.3e-09	2.6e-2	3.5e-4	3.1e-09
6	4.5e-1	7.8e-3	9.2e-09	6.4e-3	4.3e-5	1.2e-08

TABLE 3

Energy error $E_h = \|\nabla \mathbf{u} - \nabla \mathbf{u}_h\| = \|\nabla^2\varphi - \nabla^2\varphi_h\|$, the L^2-error $e_h = \|u - u_h\| = \|\nabla\varphi - \nabla\varphi_h\|$, and the difference $d_h = \|\nabla \times \varphi - \mathbf{u}\|$ for the problem with simply supported boundary and different discretization orders on a sequence of mesh refinement levels L.

	Q_2 and $RT_1 \times Q_1$			Q_3 and $RT_2 \times Q_2$		
L	E_h	e_h	d_h	E_h	e_h	d_h
2	1.5e-0	2.7e-1	7.2e-11	3.8e-1	4.4e-2	2.5e-10
3	8.0e-1	7.6e-2	3.8e-10	9.3e-2	5.4e-3	7.1e-10
4	4.0e-1	1.9e-2	1.7e-09	2.3e-2	6.5e-4	2.3e-09
5	2.0e-1	4.9e-3	6.9e-09	5.7e-3	7.9e-5	9.1e-09
6	1.0e-1	1.2e-3	2.8e-08	1.4e-3	9.8e-6	3.7e-08

On the square domain $\Omega = (-1, 1)^2$ we choose one of

$$p(t) = \begin{cases} p_c(t) = (t^2 - 1)^2, \\ p_s(t) = \frac{1}{5}t^4 - \frac{6}{5}t^2 + 1. \end{cases} \tag{3.15b}$$

It can be verified easily that p_c and p_s generate solutions with clamped/no-slip boundary and simply supported/slip boundary conditions, respectively.

In Table 2, we show errors of the discrete solutions with clamped/no-slip boundary conditions on a sequence of meshes, where level $L = 0$ corresponds to a single cell $T = (-1, 1)^2$, and each further level is generated by uniform refinement of the previous, thus yielding a mesh size $h = 2^{1-L}$. Results are reported for tensor product polynomials of degree three (Q_3) and four (Q_2) for φ_h and the matching spaces $RT_2 \times Q_2$ and $RT_1 \times Q_1$, respectively. Clearly, while the errors and the solutions are nonzero, the difference $\nabla \times \varphi - \mathbf{u}$ is on the order of machine accuracy. The errors for Q_4 and $RT_3 \times Q_3$ are not displayed and are of the magnitude of machine accuracy, as expected.

Similarly, results as in Table 2 are obtained if the clamped boundary condition is replaced by the simply supported boundary (see Table 1). In this case, we choose the solution polynomial $p_s(t)$ in (3.15b). Results for this problem with different discretization orders are presented in Table 3. Again, we see that even if the errors do not vanish, $\nabla \times \varphi = \mathbf{u}$ holds up to machine accuracy.

We close this section by considering the case of a domain which is not simply connected, namely, the example $\Omega = (-1, 1)\backslash[-0.2, 0.2]$. In this case, we expect $\nabla \times \varphi = \mathbf{u}$ only, if the right-hand side is orthogonal to harmonic vector fields in $\mathcal{H}$. Computing an analytic solution of the vector-Laplacian on Ω is a daunting task, but we can easily conclude that $\mathcal{H}$ is spanned by a vector field swirling around the origin and thus cannot be an even function in x and y. Thus, any even function in x and y is orthogonal. First, we choose $\mathbf{f}_1 = (10y^2, 0)$, which is even and thus orthogonal to $\mathcal{H}$.

TABLE 4
The difference $d_h = \|\nabla\times\varphi - \mathbf{u}\|$ on a sequence of meshes with refinement level L on a domain with a hole. The right-hand side $\mathbf{f}_1$ is orthogonal to $\mathcal{H}$, the right-hand side $\mathbf{f}_2$ is not. Refer also to Figure 1.

	Q_2 and $RT_1 \times Q_1$		Q_3 and $RT_2 \times Q_2$	
L	$\mathbf{f}_1$	$\mathbf{f}_2$	$\mathbf{f}_1$	$\mathbf{f}_2$
2	7.4e-11	0.1351	8.1e-11	0.1280
3	2.4e-10	0.1302	2.7e-10	0.1258
4	9.5e-10	0.1271	1.0e-09	0.1244
5	2.8e-09	0.1252	4.0e-09	0.1235

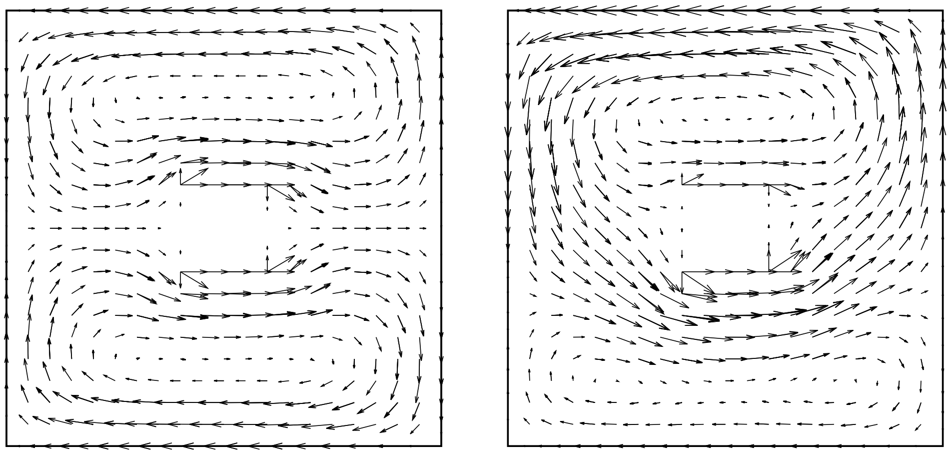

FIG. 1. *The vector fields $\mathbf{u}_h$ on a not simply connected domain with right-hand sides $\mathbf{f}_1$ (left) and $\mathbf{f}_2$ (right).*

Then, we choose $\mathbf{f}_2 = \mathbf{f}_1 + (0, x)$, which is not. Clearly, both functions are divergence free. As boundary conditions, we choose slip and simply supported, respectively. In Table 4, we show that, as expected, the difference $d_h = \|\nabla\times\varphi - \mathbf{u}\|$ is on the order of machine accuracy with right-hand side $\mathbf{f}_1$, while it converges to a nonzero number for $\mathbf{f}_2$.

In Figure 1, we show the discrete solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ for the two right-hand sides $\mathbf{f}_1$ and $\mathbf{f}_2$, respectively. The solution $\mathbf{u}_1$ exhibits symmetry with respect to the horizontal center line with no circulation around the hole. On the other hand, the solution $\mathbf{u}_2$ on the right clearly shows such a circulation and is not symmetric. When $\mathbf{f}_2$ is used as a right-hand side in the biharmonic problem (3.11), we obtain the vector field on the left of Figure 2 as $\nabla\times\varphi_2$. As expected, this vector field exhibits the same symmetry properties as $\mathbf{u}_1$, since it is orthogonal to $\mathcal{H}_h$. Thus, it cannot approximate $\mathbf{u}_2$. The difference between the two is shown on the right in Figure 2. Without being able to verify this analytically, the vector field there at least optically resembles what we would expect an element in $\mathcal{H}_h$ to look like.

4. Applications of the results. The equivalence Theorem 3.2 and its Corollary 3.3 enable us to transfer mathematical results between the biharmonic problem and the Stokes problem, thus enriching the repertoire for each of them. First, we use the fact that error estimates have been proved for the $\mathcal{C}^0$-IP method to obtain velocity estimates for the Stokes problem, which do not involve the pressure solution.

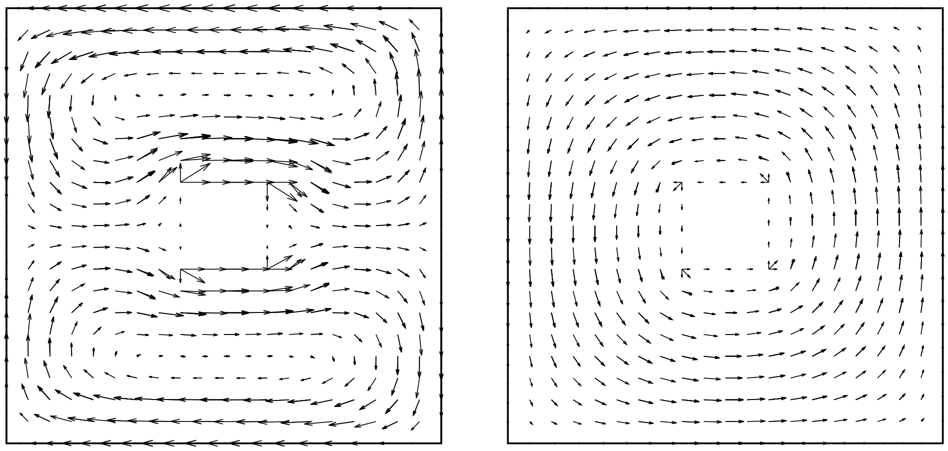

FIG. 2. *The vector field $\nabla\times\varphi_h$ on a not simply connected domain with right-hand side $\mathbf{f}_2$ (left) and the difference $u_h - \nabla\times\varphi_h$ (right).*

Then, we derive a discrete second order Korn inequality for the fourth order problem (2.10). Finally, we use multigrid results for the Stokes problem to derive uniform preconditioners for the biharmonic problem with clamped boundary conditions.

4.1. A posteriori error estimates. The stream function formulation for the Stokes problem enables us in particular to derive an error estimate for the Stokes velocity, which does not involve the pressure. To this end, we recall the estimate for the biharmonic problem from [26], applied to solutions to (3.11). Let

$$\eta_{T,B}^2 = h_T^4 \big\| \nabla\times\mathbf{f} - \Delta^2\varphi_h \big\|_T^2, \tag{4.1}$$

$$\eta_{F,B}^2 = h_F^3 \big\| [\![\partial_n \Delta\varphi_h]\!] \big\|_F^2 + h_F \big\| [\![\partial_n^2 \varphi_h]\!] \big\|_F^2. \tag{4.2}$$

Then, the estimate

$$\big\| \varphi - \varphi_h \big\|_{2,h}^2 \le \eta_B^2 \tag{4.3}$$

holds, where

$$\eta_B^2 := \sum_{T\in\mathbb{T}_h} \eta_{T,B}^2 + \sum_{F\in\mathbb{F}_h} \eta_{F,B}^2.$$

A residual-based Stokes estimator in $\mathbf{V}_h$ has been analyzed in [31, 35] and is defined as

$$\eta_S^2 := \sum_{T\in\mathbb{T}_h} h_T^2 \big\| f + \Delta u_h - \nabla p_h \underline{I} \big\|_T^2 + \sum_{F\in\mathbb{F}_h} \Big(h_F \big\| [\![p_h\underline{I} - \nabla u_h]\!] . \mathbf{n} \big\|_F^2 + \big\| \gamma_h \, [u_h] \big\|_F^2 \Big),$$

where $\underline{I}$ denotes the 2×2 identity matrix.

The reliability of this estimator has been established in [31, 35] and there holds

$$\big\| \mathbf{u} - \mathbf{u}_h \big\|_{1,h}^2 + \big\| p - p_h \big\|^2 \le C\, \eta_S^2.$$

It is a well-known result (see [36]) that the jump term $\sum_{F\in\mathbb{F}_h} \left\|\gamma_h \left[u_h\right]\right\|_F^2$ appearing in η_S is bounded above by the estimator η_S itself, provided that γ_h is sufficiently large. Hence for implementation purposes, we consider the following form of the estimator $\eta_{S,\mathbf{V}_h}$:

$$\eta_{S,\mathbf{V}_h}^2 := \sum_{T\in\mathbb{T}_h} h_T^2 \left\|f + \Delta u_h - \nabla p_h \underline{I}\right\|_T^2 + \sum_{F\in\mathbb{F}_h} \Big(h_F \left\| [\![p_h \underline{I} - \nabla u_h]\!] .\mathbf{n} \right\|_F^2 \Big). \tag{4.4}$$

The above estimator is derived from the residual

$$\begin{aligned} \mathcal{R}(\mathbf{v}) &:= \sum_{T\in\mathbb{T}_h} \mathcal{R}_T(\mathbf{v}) = 0, \\ \text{where} \quad \mathcal{R}_T(\mathbf{v}) &:= (\mathbf{f}, \mathbf{v})_T - (\nabla \mathbf{u}_h, \nabla \mathbf{v})_T + (p_h, \nabla\cdot\mathbf{v})_T. \end{aligned} \tag{4.5}$$

If we restrict $\mathbf{v}$ to $\mathbf{V}_h^0 + \mathbf{V}^0$, the last term involving the pressure naturally vanishes, thus resulting in the following estimator, which is independent of pressure:

$$\eta_{S,\mathbf{V}_h^0}^2 := \sum_{T\in\mathbb{T}_h} h_T^2 \left\|f + \Delta u_h\right\|_T^2 + \sum_{F\in\mathbb{F}_h} \Big(h_F \left\| [\![\nabla u_h]\!] .\mathbf{n} \right\|_F^2 \Big). \tag{4.6}$$

Furthermore, in (4.5), if we explicitly write $\mathbf{v} = \nabla\times\psi$ with $\psi \in \Psi_h + \Psi$, we obtain the following representation of $\mathcal{R}_T(.)$:

$$\begin{aligned} \mathcal{R}_T(\mathbf{v}_h) &= (\mathbf{f}, \nabla\times\psi) + (\Delta\mathbf{u}_h, \nabla\times\psi) - \sum_{F\in\partial T} \langle \partial_n \mathbf{u}_h, \nabla\times\psi \rangle_F \\ &= (\mathbf{f}, \nabla\times\psi) + (\nabla\times\Delta\mathbf{u}_h, \psi) + \sum_{F\in\partial T} \langle \mathbf{n}\times\Delta\mathbf{u}_h, \psi \rangle_F \\ &\quad - \sum_{F\in\partial T} \langle \partial_n \mathbf{u}_h, \nabla\times\psi \rangle_F \\ &= (\nabla\times(\mathbf{f} + \Delta\mathbf{u}_h), \psi) + \sum_{F\in\partial T} \langle \mathbf{n}\times\Delta\mathbf{u}_h + \nabla\times\partial_n\mathbf{u}_h, \psi \rangle_F . \end{aligned} \tag{4.7}$$

Thus, on introducing the element and face residuals,

$$\eta_T^2 = h_T^4 \left\|\nabla\times(\mathbf{f} + \Delta\mathbf{u}_h)\right\|_T^2, \tag{4.8}$$

$$\eta_F^2 = h_F^3 \left\| [\![\mathbf{n}\times\Delta\mathbf{u}_h]\!] \right\|_F^2 + h_F \left\| [\![\partial_n \mathbf{u}_h]\!] \right\|_F^2, \tag{4.9}$$

the residual-type estimator $\eta_{S,\mathrm{curl}}$ corresponding to the divergence-free weak formulation of the Stokes problem assumes the form

$$\eta_{S,\mathrm{curl}}^2 = \sum_{T\in\mathbb{T}_h} \eta_T^2 + \sum_{F\in\mathbb{F}_h} \eta_F^2. \tag{4.10}$$

COROLLARY 4.1. *If* φ_h *solves* (3.11) *and* $\mathbf{u}_h$ *solves* (3.6), *then the estimators* η_B *and* $\eta_{S,\mathrm{curl}}$ *are equal up to a constant.*

Proof. By expressing $\mathbf{u}_h = \nabla\times\varphi_h$, and $\mathbf{v} = \nabla\times\psi$, the error equation in (4.5) can also be written as

Table 5

The robustness of the velocity estimator $\eta_{S,\mathbf{V}_h^0}$ tested for widely ranging choices of the pressure for different levels of uniform refinement L using $RT_1 \times Q_1$ approximation spaces.

L	η_{S,V_h^0} (with no-slip boundary) $\alpha = 10$	$\alpha = 10^2$	$\alpha = 10^3$	$\alpha = 10^4$	Efficiency Index $\frac{\lVert\nabla\mathbf{u}-\nabla\mathbf{u}_h\rVert}{\eta_{S,V_h^0}}$
0	4.3e+1	4.3e+1	4.3e+1	4.3e+1	5.7
1	2.0e+1	2.0e+1	2.0e+1	2.0e+1	5.0
2	9.3e+0	9.3e+0	9.3e+0	9.3e+0	4.7
3	4.5e+0	4.5e+0	4.5e+0	4.5e+0	4.6
4	2.2e+0	2.2e+0	2.2e+0	2.2e+0	4.7
5	1.1e+0	1.1e+0	1.1e+0	1.1e+0	4.8
L	η_{S,V_h^0} (with slip boundary) $\alpha = 10$	$\alpha = 10^2$	$\alpha = 10^3$	$\alpha = 10^4$	Efficiency Index $\frac{\lVert\nabla\mathbf{u}-\nabla\mathbf{u}_h\rVert}{\eta_{S,V_h^0}}$
0	1.1e+1	1.1e+1	1.1e+1	1.1e+1	7.0
1	4.7e+0	4.7e+0	4.7e+0	4.7e+0	5.7
2	2.1e+0	2.1e+0	2.1e+0	2.1e+0	5.1
3	1.0e+0	1.0e+0	1.0e+0	1.0e+0	5.0
4	5.0e-1	5.0e-1	5.0e-1	5.0e-1	5.0
5	2.5e-1	2.5e-1	2.5e-1	2.5e-1	5.0

$$
\begin{aligned}
\mathcal{R}_T(\mathbf{v}_h^0) &= (\nabla\times\mathbf{f},\psi) - (\nabla(\nabla\times\varphi_h),\nabla(\nabla\times\psi)) \\
&= (\nabla\times\mathbf{f},\psi) - (\nabla^2\varphi_h,\nabla^2\psi) \\
&= (\nabla\times\mathbf{f},\psi) + (\nabla.\nabla^2\varphi_h,\nabla\psi) - \sum_{F\in\partial T}\left\langle\partial_n^2\varphi_h,\nabla\psi\right\rangle_F \\
&= (\nabla\times\mathbf{f},\psi) - (\Delta^2\varphi_h,\psi) + \sum_{F\in\partial T}\Big(\left\langle\partial_n\Delta\varphi_h,\psi\right\rangle_F - \left\langle\partial_n^2\varphi_h,\nabla\psi\right\rangle_F\Big).
\end{aligned}
$$

This allows us to derive the biharmonic element and face residuals as described in (4.1) and so we can conclude the equivalence of the two estimators. □

As a consequence of this equivalence, we can take advantage of the existing reliability of the estimator η_B proved in [26] for higher order finite elements, to conclude the reliability of the estimator η_{S,V_h^0} independent of the pressure.

We verify this result by numerical experiments for the Stokes problem with homogeneous no-slip and slip boundary conditions on the computational domain $\Omega = (-1,1)^2$ and with $\mathbf{f}$ defined as

$$\mathbf{f} = \mathbf{f}^0 + \alpha\nabla\underline{P},$$

where the divergence-free component $\mathbf{f}^0$ of $\mathbf{f}$ is as described in (3.15) for slip and no-slip boundary conditions. Furthermore, $\underline{P} = (x^2-x)(y^2-y)$.

For different choices of α, we report the performance of the estimator $\eta_{S,\mathbf{V}_h^0}$ in Table 5. As expected, the estimates are independent of α and thus of the component of $\mathbf{f}$ orthogonal to $\mathbf{V}_0$. Thus, we report efficiency indices measured by the ratio of the estimator $\eta_{S,\mathbf{V}_h^0}$ and the discretization error $E_h = \lVert\nabla\mathbf{u}-\nabla\mathbf{u}_h\rVert$ only for the case $\alpha = 10^4$. They stay close to 5 for both the no-slip and slip boundary conditions. For comparison, results for $\eta_{S,\mathbf{V}_h}$ are shown in Table 6. Clearly, this estimator cannot separate velocity and pressure errors and thus becomes suboptimal if only the velocity error is of concern.

TABLE 6

The Stokes estimator $\eta_{S,\mathbf{V}_h}$ as described in (4.4) computed using approximations based on $RT_1 \times Q_1$. The increasing values of the pressure proportionally impact the value of this pressure-dependent estimator over the different levels L of uniform mesh refinement.

	$\eta_{S,\mathbf{V}_h}$ (with no-slip boundary)			
L	$\alpha = 10$	$\alpha = 10^2$	$\alpha = 10^3$	$\alpha = 10^4$
0	4.5e+1	1.1e+2	9.9e+2	9.9e+3
1	2.0e+1	5.4e+1	4.9e+2	4.7e+3
2	9.6e+0	2.6e+1	2.5e+2	2.5e+3
3	4.7e+0	1.3e+1	1.2e+2	1.2e+3
4	2.3e+0	6.6e+0	6.2e+1	6.2e+2
5	1.1e+0	3.3e+0	3.1e+1	3.1e+2
	$\eta_{S,\mathbf{V}_h}$ (with slip boundary)			
L	$\alpha = 10$	$\alpha = 10^2$	$\alpha = 10^3$	$\alpha = 10^4$
0	1.5e+1	1.1e+2	9.9e+2	9.9e+3
1	6.9e+0	5.0e+1	4.9e+2	4.9e+3
2	3.2e+0	2.5e+1	2.5e+2	2.5e+3
3	1.6e+0	1.2e+1	1.2e+2	1.2e+3
4	7.9e-1	6.2e+0	6.2e+1	6.2e+2
5	3.9e-1	3.1e+0	3.1e+1	3.1e+2

A note on the implementation: since we are approximating in $RT_1 \times Q_1$, the element residual η_T corresponding to the estimator $\eta_{S,\mathbf{V}_h^0}$ in (4.8) reduces to $\eta_T^2 = h_T^4 \left\|\nabla\times\mathbf{f}\right\|_T^2$ while the face residuals remain unchanged.

4.2. The symmetric gradient. The same computations as above can be applied to the formulation (2.3), yielding a $\mathcal{C}^0$-IP method for the fourth order problem (2.10). Indeed, the interior penalty formulation for the Stokes problem with symmetric gradients has already been employed in [34, 27]. It reads as follows: find $(\mathbf{u}, p) \in \mathbf{V}_h \times Q_h$ such that for all test functions $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$ there holds

$$\tilde{a}_h(\mathbf{u},\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) - (q, \nabla\cdot\mathbf{u}) = (f, v)_\Omega\,, \tag{4.11}$$

where we employed the bilinear form defined by

$$\begin{aligned} 2\tilde{a}_h(\mathbf{u},\mathbf{v}) = (\mathbf{D}\mathbf{u}, \mathbf{D}\mathbf{v})_{\mathbb{T}_h} + a_p^i(\mathbf{u},\mathbf{v}) + a_p^\partial(\mathbf{u},\mathbf{v}) - \tfrac{1}{2}\left\langle[\![\mathbf{D}u]\!], [\![\mathbf{v}\otimes\mathbf{n}]\!]\right\rangle_{\mathbb{F}_h^i} \\ - \tfrac{1}{2}\left\langle[\![\mathbf{u}\otimes\mathbf{n}]\!], [\![\mathbf{D}v]\!]\right\rangle_{\mathbb{F}_h^i} - \left\langle\mathbf{D}u, \mathbf{v}\otimes\mathbf{n}\right\rangle_{\mathbb{F}_h^\partial} - \left\langle\mathbf{u}\otimes\mathbf{n}, \mathbf{D}v\right\rangle_{\mathbb{F}_h^\partial}. \end{aligned} \tag{4.12}$$

Entering $\mathbf{u} = \nabla\times\varphi$ and $\mathbf{v} = \nabla\times\psi$ with $\varphi, \psi \in \Psi_h$, we obtain a fourth order bilinear form

$$\begin{aligned} \tilde{b}_h(\varphi,\psi) = (\mathbf{D}\nabla\times\varphi, \mathbf{D}\nabla\times\psi)_{\mathbb{T}_h} + b_p^i(\varphi,\psi) + b_p^\partial(\varphi,\psi) \\ - \tfrac{1}{2}\left\langle[\![\mathbf{D}\nabla\times\varphi]\!], [\![\nabla\times\psi\times\mathbf{n}]\!]\right\rangle_{\mathbb{F}_h^i} - \tfrac{1}{2}\left\langle[\![\nabla\times\varphi\times\mathbf{n}]\!], [\![\mathbf{D}\nabla\times\psi]\!]\right\rangle_{\mathbb{F}_h^i} \\ - \tfrac{1}{2}\left\langle\mathbf{D}\nabla\times\varphi, \nabla\times\psi\times\mathbf{n}\right\rangle_{\mathbb{F}_h^\partial} - \tfrac{1}{2}\left\langle\nabla\times\varphi\times\mathbf{n}, \mathbf{D}\nabla\times\psi\right\rangle_{\mathbb{F}_h^\partial}. \end{aligned} \tag{4.13}$$

The resulting weak formulation is as follows: find $\varphi \in \Psi_h$ such that $\forall\psi \in \Psi_h$ there holds

$$\tilde{b}_h(\varphi,\psi) = (\mathbf{f}, \nabla\times\psi)\,. \tag{4.14}$$

This form was already discussed in [8] in the context of domain decomposition methods. Since we replaced the gradient by the symmetric gradient $\mathbf{D}$, we have to discuss stability of this method, as in the following corollary.

Corollary 4.2. *If the boundary conditions imposed on the space Ψ_h match those of the space $\mathbf{V}_h$ in the sense of* (2.13)*, and if a Korn inequality*

$$\|\mathbf{v}\|_{1,h} \le c\sqrt{\tilde{a}_h(\mathbf{v},\mathbf{v})}$$

holds for all $\mathbf{v} \in \mathbf{V}_h$, then there holds for all functions $\psi \in \Psi_h$ the second order Korn inequality

$$b_h(\psi,\psi) \le c\,\tilde{b}_h(\psi,\psi). \tag{4.15}$$

Different versions of Korn's inequality transfer accordingly.

4.3. Multigrid and domain decomposition. Efficient multigrid methods for the Stokes formulation (3.6) have been discussed recently in [33]. In view of our results here, these methods exhibit the important features that

1. they are multigrid methods performed on the product spaces $\mathbf{V}_\ell \times Q_\ell$ for different refinement levels ℓ, and
2. the smoothers rely on an overlapping subspace decomposition into $\mathbf{V}_\ell^i \times Q_\ell^i$ which in turn can be completed by a space Ψ_ℓ^i to a sequence like (2.7).

Thus, we can use the equivalence result to obtain a multigrid smoother for the biharmonic problem, in particular with clamped boundary conditions. The outline of the method is as follows. First, instead of a single discrete velocity space $\mathbf{V}_h$ in the previous sections, we consider a hierarchy of spaces $\{\mathbf{V}_\ell\}_{\ell=0,\dots,L}$. The index h for finite element spaces and bilinear forms is replaced in this section by the index ℓ for the discrete spaces and forms on level ℓ. Then, instead of a simple hierarchy of finite element spaces, we introduce a hierarchy of cochain complexes,

$$\begin{array}{ccccc}
I_{\Psi,\ell-1}\big\downarrow & & I_{\mathbf{V},\ell-1}\big\downarrow & & I_{Q,\ell-1}\big\downarrow \\
\Psi_{\ell-1} & \xrightarrow{\nabla\times} & \mathbf{V}_{\ell-1} & \xrightarrow{\nabla\cdot} & Q_{\ell-1} \\
I_{\Psi,\ell}\big\downarrow & & I_{\mathbf{V},\ell}\big\downarrow & & I_{Q,\ell}\big\downarrow \\
\Psi_{\ell} & \xrightarrow{\nabla\times} & \mathbf{V}_{\ell} & \xrightarrow{\nabla\cdot} & Q_{\ell} \\
I_{\Psi,\ell+1}\big\downarrow & & I_{\mathbf{V},\ell+1}\big\downarrow & & I_{Q,\ell+1}\big\downarrow
\end{array}, \tag{4.16}$$

with level spaces Ψ_ℓ, $\mathbf{V}_\ell$, and Q_ℓ and embedding operators I_* such that all loops in the diagram (4.16) commute. While [33] was concerned with multigrid cycles for the hierarchy $\{\mathbf{V}_\ell \times Q_\ell\}$, more precisely, for the hierarchy of the divergence-free subspaces $\{\mathbf{V}_\ell^0\}$, here we consider the hierarchy $\{\Psi_\ell\}$. But due to the equivalence, the two are isomorphic on simply connected domains, and solutions to the residual equations on each level are related by Theorem 3.2. Thus, it remains to argue that the smoother used in [33] can be lifted from $\mathbf{V}_\ell \times Q_\ell$ into the space Ψ_ℓ.

The overlapping subspace smoother for the Stokes problem in [33] is based on methods found in [4, 5, 39, 40] as well as [1, 2, 25] and constructed the following way: for each vertex P_ℓ^i of the mesh $\mathbb{T}_\ell$ on level ℓ, let Ω_ℓ^i be the patch of all cells which contain this vertex. Then, $\mathbf{V}_\ell^i$ and Q_ℓ^i are spanned by the functions in $\mathbf{V}_\ell$ and Q_ℓ with support in Ω_ℓ^i. The subspace solvers then correspond to solving a Stokes problem (3.6) in the interior of Ω_ℓ^i with no-slip boundary conditions. They can be written as projection operators $T_\ell^i : \mathbf{V}_\ell \times Q_\ell \to \mathbf{V}_\ell^i \times Q_\ell^i$ such that $T_\ell^i(\mathbf{u}_\ell, p_\ell)$ is the unique solution $(\mathbf{w}, r) \in \mathbf{V}_\ell^i \times Q_\ell^i$ of the problem

$$
\begin{aligned}
(4.17)\qquad & a_\ell(\mathbf{w},\mathbf{v}) - (r,\nabla\cdot\mathbf{v}) - (q,\nabla\cdot\mathbf{w}) \\
&= a_\ell(\mathbf{u}_\ell,\mathbf{v}) - (p_\ell,\nabla\cdot\mathbf{v}) - (q,\nabla\cdot\mathbf{u}_\ell) \quad \forall(\mathbf{v},q)\in\mathbf{V}_\ell^i\times Q_\ell^i.
\end{aligned}
$$

By construction, the velocity solution $\mathbf{w}$ in the equation above is in the divergence-free subspace $\mathbf{V}_\ell^{0;i}\subset\mathbf{V}_\ell^i$, such that we can define the projection operator

$$
\breve{T}_\ell^i : \mathbf{V}_\ell^0 \to \mathbf{V}_\ell^{0;i} \tag{4.18}
$$

as the restriction of T_ℓ^i to $\mathbf{V}_\ell^0$.

In order to develop a smoothing technique for the $\mathcal{C}^0$-IP method, for each Ω_ℓ^i let the space Ψ_ℓ^i be the subspace of functions in Ψ_ℓ with support in Ω_ℓ^i. This definition induces the following commutative diagram:

$$
\begin{array}{ccccc}
\Psi_\ell^i & \xrightarrow{\nabla\times} & \mathbf{V}_\ell^i & \xrightarrow{\nabla\cdot} & Q_\ell^i \\
{\scriptstyle I_{\Psi,\ell}^i}\big\downarrow & & {\scriptstyle I_{\mathbf{V},\ell}^i}\big\downarrow & & {\scriptstyle I_{Q,\ell}^i}\big\downarrow \\
\Psi_\ell & \xrightarrow{\nabla\times} & \mathbf{V}_\ell & \xrightarrow{\nabla\cdot} & Q_\ell
\end{array}. \tag{4.19}
$$

Using (4.19) and applying Theorem 3.2, we can substitute curls for velocities in (4.17) to define a new projection operator $P_\ell^i:\Psi_\ell\to\Psi_\ell^i$ through

$$
b_\ell(P_\ell^i\varphi_\ell,\psi) = b_\ell(\varphi_\ell,\psi) \quad \forall\psi\in\Psi_\ell^i, \tag{4.20}
$$

which amounts to replacing the local Stokes problem (4.17) on each patch Ω_ℓ^i by a biharmonic problem (3.11) on the same patch with clamped boundary conditions. By the arguments above, the projection operators $\breve{T}_\ell^i$ and P_ℓ^i have the following commutative diagram property:

$$
\begin{array}{ccc}
\Psi_\ell & \xrightarrow{\nabla\times} & \mathbf{V}_\ell^0 \\
{\scriptstyle P_\ell^i}\big\downarrow & & {\scriptstyle \breve{T}_\ell^i}\big\downarrow \\
\Psi_\ell^i & \xrightarrow{\nabla\times} & \mathbf{V}_\ell^{0;i}
\end{array}. \tag{4.21}
$$

Thus, we obtain the next corollary.

COROLLARY 4.3. *A variable V-cycle method for the biharmonic problem, using standard embedding and projection operators between the level spaces Ψ_ℓ, and Schwarz-type smoothers solving biharmonic problems on the interior of vertex patches provides a uniform preconditioner for the biharmonic problem* (3.11).

The implementation of such a smoother is particularly simple, because the subspace solver corresponds to inverting a diagonal block of the system matrix. As with all subspace correction methods, we have the choice between the additive (Jacobi) and the multiplicative (Gauss–Seidel) version. In the example below, we decided for the symmetric multiplicative version, where we traverse the list of vertex patches twice for one smoothing step, once in a given ordering and once in reverse. According to the commuting diagrams (4.16), (4.19), and (4.21), the smoother for the Stokes and biharmonic problems should perform with the same efficiency.

These considerations are confirmed in Table 7, where we compare iteration counts for the biharmonic and the Stokes problem with clamped and no-slip boundary conditions, respectively. We start at refinement level $L = 3$, corresponding to 8×8 mesh cells, and compare the number of GMRES steps to reduce the initial residual

Table 7

GMRES iterations to reduce the initial residual by a factor 10^{10} using multigrid preconditioning.

	Biharmonic			Stokes		
L	Q_2	Q_3	Q_4	RT_1	RT_2	RT_3
3	6	7	6	7	7	6
4	12	8	7	12	8	7
5	15	8	7	15	8	7
6	16	9	7	16	9	7
7	17	9	7	17	9	
8	18	9	7	18		

by 10^{10}. Biquadratic, bicubic, and biquartic polynomials are used for Ψ_h, and the corresponding Raviart–Thomas elements are used for $\mathbf{V}_h$. The spaces Q_h are chosen in a matching way. The results clearly exhibit the close similarity of the behavior of the two methods. Comparing these results to the methods studied by Zhao in [47, 48, 49], we see that while our smoother involves more effort than standard smoothers, our method competes by using only a single pre- and postsmoothing step on the finest level and very fast convergence with average contraction rates below 0.1 for higher order elements.

4.4. Lifting operators. In the analysis of DG methods, the introduction of lifting operators can significantly simplify and unify the analysis of whole classes of methods [37, 41]. They simplify the analysis by replacing the terms $a_c^i(.,.)$ in (3.1) by an alternative term, which is bounded in H^1. Thus, no further regularity is required to enter a continuous solution into the discrete form. (Regularity is needed for consistency, though.)

The purpose of this section is to establish a relationship between the lifting operators corresponding to the Stokes and the biharmonic problem.

We shortly briefly DG lifting operators. To this end, we first introduce a discrete auxiliary space

$$\Sigma_h := \left\{\tau \in L^2(\Omega, \mathbb{R}^{2\times 2}) \middle| \tau|_T \in \mathcal{P}_k(T)^{2\times 2} \quad \forall T \in \mathbb{T}(\Omega)\right\},$$

where $\mathcal{P}_k(T)$ is chosen such that the condition

$$\nabla \mathbf{V}_h \subset \Sigma_h \qquad \text{or} \qquad \mathbf{D} V_h \subset \Sigma_h$$

holds true, depending on whether formulation (3.6) or (4.11) is used. We focus on (3.6) here and remark that results carry over to the symmetric gradient. We define the lifting operator $\mathcal{L}_S$ through

$$\mathcal{L}_S : \mathbf{V} + \mathbf{V}_h \to \Sigma_h, \qquad (\mathcal{L}_S \mathbf{v}, \tau) = \sum_F \langle [\![\tau]\!], [\![\mathbf{v} \otimes \mathbf{n}]\!] \rangle_F . \tag{4.22}$$

It is established in [37, 41] that this operator $\mathcal{L}_S$ admits the upper bound:

$$\left\|\mathcal{L}_S(\mathbf{v})\right\|_{0,\Omega}^2 \le C \sum_{F \in \mathcal{F}_h} h_F^{-1} \left\|[\mathbf{v}]\right\|_{0,F}^2, \quad \mathbf{v} \in \mathbf{V} + \mathbf{V}_h,$$

where $C > 0$ depends on the shape regularity of the triangulation.

Analogously, we can introduce a lifting operator for the biharmonic problem through the definition

$$\mathcal{L}_{\tilde{B}} : \Psi + \Psi_h \to \Sigma_h, \qquad \mathcal{L}_{\tilde{B}}(\psi) = \mathcal{L}_S(\nabla \times \psi). \tag{4.23}$$

Since the jump of tangential velocities vanishes, letting $\mathbf{v} = \nabla\times\psi$ and $\mathbf{v}_t = (0, v_t)^T$, the definition in (4.22) is equivalent to

$$(\mathcal{L}_S\mathbf{v}, \tau) = \sum_F \langle [\![\tau]\!], [\![\mathbf{v}_t \otimes \mathbf{n}]\!]\rangle_F = \sum_F \langle [\![\tau]\!], [\![\partial_n\psi]\!]\rangle_F = (\mathcal{L}_{\tilde{B}}\psi, \tau).$$

This enables us to rewrite the consistency terms appearing in (3.1) and (3.2) as

$$a_c^i(\mathbf{u}, \mathbf{v}) + a_c^\partial(\mathbf{u}, \mathbf{v}) = \frac{1}{2}(\mathcal{L}_S\mathbf{v}, \nabla\mathbf{u})$$

and those described in (3.11) as

$$b_c^i(\varphi, \psi) + b_c^\partial(\varphi, \psi) = \frac{1}{2}(\mathcal{L}_{\tilde{B}}\psi, \nabla(\nabla\times\varphi))$$

without losing the equivalence of the two methods. Furthermore, by its definition, $L_{\tilde{B}}$ admits an upper bound in terms of $\sum_{F\in\mathcal{F}_h} h_F^{-1} \left\|[\nabla\psi]\right\|_{0,F}^2$.

The lifting $L_{\tilde{B}}$ is also related to the lifting operator $\mathcal{L}_B$ for the $\mathcal{C}^0$-IP method introduced in [26] and this is described in the corollary below:

COROLLARY 4.4. *The lifting operator $\mathcal{L}_{\tilde{B}}$ defined in* (4.23) *is equivalent to the lifting operator $\mathcal{L}_B$ in the following sense:*

$$\left(\mathcal{L}_B(\nabla\psi), \nabla^2\varphi\right) = \frac{1}{2}(\mathcal{L}_S(\nabla\times\psi), \nabla(\nabla\times\varphi)).$$

The following stability estimate has been proved in [26]:

$$\left\|\mathcal{L}_B\psi\right\|^2 \le C\sum_{F\in\mathbb{F}} h_F^{-1}\left\|[\![\partial_n\psi]\!]\right\|_F^2.$$

Thus we have obtained a corresponding equivalence relationship between lifting operator for the Stokes and the biharmonic problem which in particular holds for divergence-free velocities and extends to problems in three dimensions.

Using the equivalence of the lifting operators, we obtain a local DG (LDG) version of the $\mathcal{C}^0$-IP method. Namely, we take the lifting version of the Stokes LDG form (see [37, 21]),

$$a_{\mathrm{LDG}}(\mathbf{u}, \mathbf{v}) = (\mathbf{D}\mathbf{u} - \mathcal{L}_S\mathbf{u}, \mathbf{D}\mathbf{v} - \mathcal{L}_S\mathbf{v})_{\mathbb{T}_h^S} + \left\langle\gamma_h^2 [\![\mathbf{u}]\!], [\![\mathbf{v}]\!]\right\rangle_{\Gamma_h^S},$$

and obtain

$$b_{\mathrm{LDG}}(\varphi, \psi) = \left(\nabla^2\varphi - \mathcal{L}_{\tilde{B}}\varphi, \nabla^2\psi - \mathcal{L}_{\tilde{B}}\psi\right)_{\mathbb{T}_h^S} + \left\langle\gamma_h^2 [\![\varphi]\!], [\![\psi]\!]\right\rangle_{\Gamma_h^S}.$$

From the same property of the Stokes LDG form, we infer that $b_{\mathrm{LDG}}(.,.)$ is symmetric and positive definite for any nonzero penalty γ_h.

4.5. Remarks on three-dimensional problems. In the case that $\Omega \subset \mathbb{R}^3$, the sequence (2.5) is replaced by the longer cochain complex

$$\mathbb{R} \xrightarrow{\subset} H^2(\Omega) \xrightarrow{\nabla} H^{1,\mathrm{curl}}(\Omega) \xrightarrow{\nabla\times} H^1(\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega) \longrightarrow 0, \tag{4.24}$$

where the three-dimensional space $H^{\mathrm{div}}(\Omega)$ is analogous to section 2.1. With the standard three-dimensional curl of a vector field $\nabla\times\mathbf{v}$, we define

$$H^{1,\mathrm{curl}}(\Omega) = \left\{\mathbf{v} \in H^1(\Omega; \mathbb{R}^3) \;\middle|\; \nabla\times\mathbf{v} \in L^2(\Omega; \mathbb{R}^3)\right\}.$$

This sequence is a Hilbert cochain complex, since due to [28, Corollary 3.3] every divergence-free function in $H^1(\Omega)$ has a divergence-free vector potential in $H^2(\Omega)$. On the other hand, if the potential is divergence-free, we can change the space to $H^{1,\mathrm{curl}}(\Omega)$ without affecting the norm of the potential. Furthermore, under the assumption of Ω being simply connected, the sequence is exact.

Adding boundary conditions analogous to (2.2) and (2.6), we obtain the spaces

$$\begin{aligned}\mathbf{V}^3 &= \left\{\mathbf{v} \in H^1(\Omega) \;\middle|\; \mathbf{v}_{|\Gamma_S} \cdot \mathbf{n} = 0 \;\wedge\; \mathbf{v}_{|\Gamma_N} = 0\right\},\\ \Psi^3 &= \left\{\boldsymbol{\psi} \in H^{1,\mathrm{curl}}(\Omega) \;\middle|\; (\nabla\times\boldsymbol{\psi}_t)_{|\Gamma_S} = 0 \;\wedge\; \nabla\times\boldsymbol{\psi}_{|\Gamma_N} = 0\right\},\end{aligned}$$

where the subscript t indicates the tangential components of the curl vector. For these spaces, we apply the same computations that lead from the Stokes problems (2.3) and (2.4) to the biharmonic problems (2.10) and (2.11), respectively. Here, we have to observe that the $\equiv$ in (2.11) does not hold anymore, since the three-dimensional curl has a nontrivial kernel, and we only obtain

$$(\nabla\nabla\times\varphi, \nabla\nabla\times\psi) = (\mathbf{f}, \nabla\times\psi)\,. \tag{4.25}$$

Since either (2.10) or (4.25) is only well-posed on the subspace of Ψ^3 which is orthogonal to gradients, the stream function formulation is a mixed problem itself and thus not feasible for computing solutions. Nevertheless, the Hodge decomposition (2.14) and the characterization of the divergence-free subspace $\mathbf{V}^0$ in (2.9) remain valid. Therefore, corollaries for the Stokes problem remain valid and provide analytical tools, in particular, the arguments about error estimates in 4.1.

5. Conclusions. We have derived an algebraically exact relation between $\mathcal{C}^0$-IP methods for the biharmonic equation and divergence-conforming discontinuous Galerkin methods for the Stokes equations, respectively, where the solution to the former turns out to be the stream function of the solution to the latter. Based on this relation, we derived an error estimate based on the divergence-free part of the right-hand side and not involving the pressure for the Stokes problem. Furthermore, we derived a Korn inequality and an efficient multigrid method for the biharmonic problem.

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