Numerical Analysis of Ordinary Differential Equations Exercises

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Exercise sheet 1
Until: Thursday, 28.04.2015

There will be no scores given and accumulated over the semester. Nevertheless, we strongly recommend solving the exercises. If you choose to hand in your solutions they will be annotated according to completeness and mathematical correctness. Please hand in your solution in groups of at least 2 or 3 students.
Important: Please register to one of the exercise groups in **Müsli**.

Exercise 1.1 (**Separation of Variables**)
Recapitulate the method "Separation of Variables" for obtaining analytical solutions of ODEs. Use the method to solve the following IVP:

$$\begin{aligned} u'(t) &= 2u(t), \qquad t \geqslant 1 \\ u(1) &= e. \end{aligned}$$

Remark: Prof. Rannachers lecture notes *Analysis I* explain this method very well. Also Wikipedia is a good source for basic math topics.

Exercise 1.2 (**Uniqueness of Solutions**)

a) Consider the following initial value problem:

$$u'(t) = -u(t)^2, \qquad t \geqslant 0, \qquad u(0) = -1.$$

Find a solution. Is the solution unique in the region of existence?

b) Show that $u(t) = (\frac{3}{4}t + 1)^{\frac{4}{3}}$ is a solution of the following IVP:

$$u'(t) = u(t)^{\frac{1}{4}}, \qquad t \geqslant 0, \qquad u(0) = 1.$$

Is this solution unique ?

c) Now consider the slightly changed IVP:

$$u'(t) = u(t)^{\frac{1}{4}}, \qquad t \geqslant 0, \qquad u(0) = 0.$$

Find a solution and discuss uniqueness.

Remark: You can either guess the solutions or use separation of variables.

Exercise 1.3 (**Matrixvalued Mappings**)
Let $A \in \mathbb{C}^{n\times n}$. We define the exponential function by

$$e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \lim_{k\to\infty} \sum_{k=0}^{m} \frac{1}{k!} A^k.$$

Assume absolute convergence.

a) Let $D \in \mathbb{C}^{n\times n}$ be a diagonal matrix. Calculate e^D.

b) Let $A \in \mathbb{C}^{n\times n}$ and $S \in \mathbb{C}^{n\times n}$ invertible. Show that $e^{SAS^{-1}} = Se^A S^{-1}$.

c) Consider the following IVP for vector-valued functions $u : [0, T] \to \mathbb{C}^n$

$$\begin{aligned} u' &= Au, \\ u(0) &= u_0. \end{aligned}$$

Assume that $A \in \mathbb{C}^{n\times n}$ is diagonalisable. Show that $u(t) = e^{At}u_0$ is a solution. Can we expect uniqueness?

Exercise 1.4 (**Grönwall's inequality**)
Proof the differential formulation of Grönwall's inequality.
Let I denote an Intervall of the form $[a, b]$ or $[a, \infty)$, with $b > a$. Let $w(t)$ and $b(t)$ be continuous and nonnegative functions. Furthermore, let $w(t)$ be differentiable and satisfy the differential inequality

$$w'(t) \leqslant b(t)w(t).$$

Then w is bounded:

$$w(t) \leqslant w(a) \exp\left(\int_a^t b(s)\mathrm{d}s\right), \qquad \forall t \in I.$$