Numerical Analysis of Ordinary Differential Equations Exercises

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Exercise sheet 6
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Exercise 6.1 (**Stability**)

a) Recapitulate the calculations of the stability regions of Explicit Euler and Implicit Euler.

b) Show that the trapezoidal rule is A-stable and it holds

$$S = \{z \in \mathbb{C} |\, \Re(z) \leq 0\}.$$

Exercise 6.2 (**Properties of the Theta-method II**)
Show that for $\theta \in [\frac{1}{2}, 1)$ the stability region of the theta-method is an ellipsoid in $\{z \in \mathbb{C} \mid \Re(z) \leqslant 0\}$. For the stability intervall it holds:

$$S \cap \mathbb{R} = [-2(1-2\theta)^{-1}, 0].$$

Exercise 6.3 (**Jacobian**)

a) Convert the following skalar ODE of second order into a system of order 1.

$$u''(t) = f(t, u(t), u'(t)).$$

b) Show that the Jacobian of the right hand side of this system (of order 1) has only real eigenvalues if $\partial_x f(t, x, z) \geqslant 0$.
What are the consequences regarding the approximation properties of the methods introduced in the lecture ?

Exercise 6.4 (**Dissipativity**)

A usually very important property of the solution u of an ODE is the conservation of energy. Imagine the oscillating solution of a pendulum. The numerical method should keep the amplitude of the oscillation.

A method with stability function $R(z)$ is called *weakly dissipativ* if $R(\pm\mathrm{i}) \approx 1$.

a) What is the explicit solution of the modelproblem $u'(t) = \pm\mathrm{i}u(t), u(0) = 1$?

b) Analyze the dissipativity of different methods:

   - Explicit Euler
   - Implicit Euler
   - Trapezoidal Rule

   Remark: The stability function $R(z)$ describes the behavior of the numerical solution: $u_m = R(\lambda h)u_{m-1}$, see definition 3.2.2.

c) Show that for the general Theta-Scheme it holds:

$$|R(\mathrm{i})| \approx 1, \qquad \text{if } \theta \approx 1.$$