Numerical Analysis of Ordinary Differential Equations Exercises

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Exercise sheet 8
Until: —

The following questions are an opportunity to ensure that you understood the lecture. Please don't hand in your answers.

Exercise 8.1 (**Questions I: ODE theory**)

a) Consider the following d-dimensional linear IVP:

$$u'(t) + Au(t) = b(t), \quad t \geqslant 0, \quad u(0) = u_0,$$

with a matrix $A \in \mathbb{R}^{n\times n}$ and a continuous vector-valued function $b : [0, \infty) \to \mathbb{R}^n$. Discuss existence and uniqueness of the solution $u(t)$. Is the solution bounded?

b) What is the difference between the theorems of Peano and Picard-Lindelöf?

c) Which condition guarantees local existence of a solution?

d) Which condition do you need for uniqueness of a solution?

e) What is the solution of $u'(t) = \lambda u(t), t \geqslant 0$ with $u(0) = 1$?

Exercise 8.2 (**Questions II: One-step methods**)

a) What is Grönwall's inequality and what is its use?

b) How is the truncation error of a one-step method $y_1 = y_0 + hF(h; t_0, y_1, y_0)$ defined?

c) How are convergence order and consistency order defined?
How are they connected?

d) Describe the construction of Runge-Kutta methods within a few words.

e) Describe the structure of explicit RK methods.

f) Does there exist a RK method of order 4 with 3 steps?

g) State the definiton of the trapezoidal rule.
What is the order of the trapezoidal rule?

h) Formulate the Theta-method.

i) Describe embedded RK methods and their application.

j) How large is the effort of embedded RK methods compared to regular RK methods?

k) What is the meaning of "45" in Dormand-Prince 45?

l) What is the meaning of the acronyms DIRK and SDIRK? Describe their Butcher-tableau.

m) Describe continuous RK methods.

n) What is the butcher tableau of the Theta-method?

o) What are collocation methods?

p) What are Gauß-collocation methods?

q) How are collocation methods related to RK methods?

Exercise 8.3 (**Questions III: Stability and stiffness**)

a) What is the definition of the stability region and the stability interval?

b) What are the stability regions of the explicit Euler method, implicit Euler method and trapezoidal rule?

c) Why does neither the explicit Euler nor the implicit Euler method conserve energy?

d) What is the relation of the general linear IVP $u'(t) = Au(t),\ t \geqslant 0,\ u(0) = u_0$ with $u \in \mathbb{R}^n$ and the scalar model problem $u'(t) = \lambda u(t),\ t \geqslant 0,\ u(0) = 1$?

e) What is the definition of stiffness?

f) Is an IVP with eigenvalues $\lambda = 50 \pm \mathrm{i}80$ and $\lambda = \pm 2\mathrm{i}$ stiff?

g) When is a method considered to be A-stable?

h) Name two A-stable methods.

i) Is Dormand-Prince 45 A-stable?

j) When is a method called L-stable?

k) What is the stability function of the Theta-method?

l) What is B-stability and how is it related to A-stability?

Exercise 8.4 (**Questions IV: Multistep methods**)

a) Formulate a general linear multistep method (LMM).

b) What are the first and second generating polynomials?

c) When is an LMM called stable (zero-stable / D-stable)?

d) What is the connection of stability and the generating polynomials?

e) What are the conditions of convergence for an LMM?

f) What is the definition of $A(\alpha)$-stability?

Exercise 8.5 (**Questions IV**)

a) State the Lotka-Volterra equation. What does this equation model?

b) Consider an ODE that has an oscillation as solution (e.g. $u'(t) = iu(t)$). What behavior can you expect, if you solve with the explicit Euler method or the implicit Euler method?

c) What is the purpose of adaptive refinement?

d) Describe the method "separation of variables".

e) How is e^A defined?

f) What is the definition of convergence order?

g) What is the definition of the Landau-symbol: $f(n) = \mathcal{O}(g(n)),\ n \to \infty$

h) What was the purpose of the norm $\|v\|_s = \left(\frac{1}{d}\sum_{i=1}^{d}\left(\frac{v_i}{s_i}\right)^2\right)^{\frac{1}{2}}$?