Numerical Analysis of Ordinary Differential Equations Exercises

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Exercise sheet 9
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Exercise 9.1 (BDF methods)
Consider the following stiff problem

$$\begin{aligned} u'(t) &= -10u(t) - 100v(t), \\ v'(t) &= 100u(t) - 10v(t), \\ w'(t) &= u(t) + v(t) - tw(t), \end{aligned}$$

which should be solved by a BDF method. To do this effiencently methods of higher order are prefered. Which BDF method introduced in the lecture should be chosen?

Exercise 9.2 (Switches)
Consider the following IVP with switches

$$u'(t) = \begin{cases} f_1(t,u) & \text{if } c(t,u) \geq 0 \\ f_2(t,u) & \text{else} \end{cases}, \qquad t \in (0,10), \qquad u(0) = u_0.$$

where the right hand side function switches depending on time and state.

a) Consider a switching function depending on time only, e.g. $c(u,t) = t - 7$. What is the difference when using either Runge-Kutta methods or BDF methods to solve such switching problems?

b) Consider now $c(t,u) = \sin(t)$. What are the drawbacks of self-starting BDF in this case? How could you improve your method?

c) What kind of methods are suitable to solve such problems if the switching function depends on the state, e.g. $c(t,u) = u$?

Exercise 9.3 (**Eigenvalues of the Laplace Operator**)
We want to discuss a more realistic problem than Exercise 9.1. The heat-equation has been introduced on the programming exercise sheet 03:

$$\partial_t u(t,x) - \Delta u(t,x) = 0,$$

with $\Delta = \partial_{xx}$. The solution $u(t,x)$ describes the heat distribution in time and space $(t,x) \in I \times \Omega \subset \mathbb{R} \times \mathbb{R}$. The reformulation into a system of ODEs is given by:

$$\begin{aligned} \partial_t U(t) &= -AU(t) \qquad t \geq 0 \\ U(0) &= U_0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times m}$ is a finite representation of the infinite operator $-\Delta$. If we choose $\Omega = [0,1]$ and discretise in a specific fashion, it holds:

$$A = \frac{1}{k^2} \cdot \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

with $k = \frac{1}{m+1}$. The parameters k describes the coarseness of the spatial discretisation. The smaller the stepsize k in space, the better the resolution in space, but the larger the ODE-system that has to be solved.

a) Show that A is a symmetric, positive definite Matrix. What are the consequences for the eigenvalues λ_ν ?
Remark: Use your knowledge from the lecture Numerics 0.

b) The eigenvalues of A are given by:

$$\lambda_\nu = \frac{1}{k^2}(2 - 2\cos(\nu\pi k)), \qquad \nu = 1, \ldots, m.$$

Show that $\lambda_{\text{max}} \sim \frac{1}{k^2}$.
You want to solve this ODE-System with the Explicit Euler Method on the time intervall $I = [0,1]$. Calculate the necessary amount of timesteps to get a stable solution, if $k = 10^{-3}$.