Numerical Analysis of Ordinary Differential Equations Exercises

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Exercise sheet 10
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Exercise 10.1 (**Solvability of linear BVP**)
The linear ODE of second order

$$u''(t) + u(t) = 1, \qquad t \in [a, b] \tag{10.1}$$

has the general solution

$$u(t) = A\sin(t) + B\cos(t) + 1.$$

a) Verify that there is exactly one solution for the boundary conditions $u(0) = u(\frac{\pi}{2}) = 0$, no solution for $u(0) = u(\pi) = 0$ and infinitly many solutions for $u(0) = u(\pi) = 1$. This demonstrates the difficulties of a uniform solution theory for BVP, even in the linear case.

b) Transform equation (10.1) into a system of first order ODEs and check the regularity of the Matrix $B_a + B_b U(b)$, where $U(t)$ is the fundamental matrix. This condition ensures that a linear BVP has a locally unique solution for any data $f(t)$ and g.

Exercise 10.2 (**Practical Exercise - Stiffness**)
The origin of the definiton for stiffness is based on experimental observation. Explicit Methods have a very limited stability region. Hence even smooth and simple IVPs might require a small stepsize. In comparison to that implicit methods can use large steps, but are more expensive. This observation led to the definition of stiffness introduced in the lecture, using the eigenvalues of the IVP.
In this exercise we want to observe these stepsize problems. Therefore we consider the following IVP:

$$u'(t) = \mu[u(t) - e^{-t}] - e^{-t}, \qquad \mu < 0, \qquad u(0) = 1.$$

This IVP has the exact solution $u(t) = e^{-t}$. Solve this IVP on $I = [0, 1]$ with $\mu = -100$ and $h = 2^{-i}$ for $i = 4, \ldots, 7$ with the explicit and implicit Euler method. And compare the error $\|u_n - u(1)\|$.