Finite Element Methods for Flow Simulations
- Exercise Sheet 6 -

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Exercise 11. Consider the (abstract) linear evolution problem

$$\frac{d}{dt}((u(t),v)_H + a(u(t),v) = \langle f(t),v\rangle_{(X^*,X)} \quad \forall v \in X, t \in (0,T)$$
$$u(0) = u_0.$$

where $a\colon X \times X \to \mathbb{R}$ is a

- bounded, i.e. $\exists M > 0\colon |a(u,v)| \leq M\|u\|_X\|v\|_X \quad \forall u,v \in X$, and
- X-elliptic, i.e. $\exists \gamma > 0\colon |a(u,u)| \geq \gamma\|u\|_X^2 \quad \forall u \in X$,

bilinear form. Moreover, assume $f \in L^2(0,T;X^*)$ and $u_0 \in H$ where the spaces X, H and X^* form a Gelfand triple (with inclusion constant C_I). Under these assumptions, any solution $u \in W^{1,2}(0,T;X)$ to the linear evolution problem satisfies

1. $\|u\|_{C(0,T;H)} \leq \|u_0\|_H \exp\left(-\frac{\gamma}{2C_I^2}T\right) + \frac{1}{\sqrt{\gamma}}\|f\|_{L^2(0,T;X^*)}$
2. $\|u\|_{L^2(0,T;X)} \leq \frac{1}{\sqrt{\gamma}}\|u_0\|_H + \frac{1}{\gamma}\|f\|_{L^2(0,T;X^*)}$
3. $\|u'\|_{L^2(0,T;X^*)} \leq \frac{\sqrt{2}M}{\sqrt{\gamma}}\|u_0\|_H + \frac{\sqrt{2}}{\gamma}\sqrt{\gamma^2+M^2}\|f\|_{L^2(0,T;X^*)}$

Hint: For (i) and (ii) show the inequality

$$\frac{d}{dt}\|u(t)\|_H^2 \leq -\gamma\|u(t)\|_X^2 + \frac{1}{\gamma}\|f(t)\|_X^2$$

and apply the Gronwall lemma. For (iii) show the inequality

$$\|u'(t)\|_{X^*} \leq \|f(t)\|_{X^*} + M\|u(t)\|_X$$

and apply (ii).