Augmented Taylor-Hood Elements for Incompressible Flow

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The Stationary Stokes Equations

Find velocity $\mathbf{u}$ and pressure p in a polyhedral domain $\Omega \subset \mathbb{R}^d$:

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega \end{aligned}$$

Variational formulation

Find $(\mathbf{u}, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ such that

$$\begin{aligned} \nu(\nabla\mathbf{u}, \nabla\mathbf{v})_{L^2(\Omega)} - (p, \operatorname{div}\mathbf{v})_{L^2(\Omega)} &= (\mathbf{f}, \mathbf{v})_{L^2(\Omega)} && \forall \mathbf{v} \in \mathbf{V} = H_0^1(\Omega)^d \\ (\operatorname{div}\mathbf{u}, q)_{L^2(\Omega)} &= 0 && \forall q \in Q = L_0^2(\Omega) \end{aligned}$$

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The Discretized Stokes Equations

- $\mathcal{T}_h$ regular decomposition of Ω
- $\mathbf{V}_h \times Q_h \subset V \times Q$ finite element subspace

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$:

$$\begin{aligned} \nu(\nabla \mathbf{u_h}, \nabla \mathbf{v_h})_{L^2(\Omega)} - (p_h, \operatorname{div} \mathbf{v_h})_{L^2(\Omega)} &= (\mathbf{f}, \mathbf{v_h})_{L^2(\Omega)} & \forall \mathbf{v_h} \in \mathbf{V_h} \\ (\operatorname{div} \mathbf{u_h}, q_h)_{L^2(\Omega)} &= 0 & \forall q_h \in Q_h \end{aligned}$$

Remark

- Exact mass conservation is fulfilled if $\operatorname{div} \mathbf{V_h} \subset Q_h$.
- For the Taylor-Hood element just $\int_\Omega \operatorname{div} \mathbf{u_h}\, dx = 0$ can be expected.

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The Inf-Sup Condition

Theorem

The discretized Stokes problem has exactly one solution, if the spaces $\mathbf{V}_h$ and Q_h satisfy the inf-sup condition

$$\exists \beta > 0, \beta \neq \beta(h) \textit{ where}$$

$$\inf_{\substack{q_h \in Q_h \\ q_h \neq 0}} \sup_{\substack{\mathbf{v_h} \in \mathbf{V_h} \\ \mathbf{v_h} \neq 0}} \frac{\int_\Omega q_h \operatorname{div} v_h \, dx}{\|\mathbf{v_h}\|_\mathbf{V} \|q_h\|_Q} \geq \beta.$$

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- The Taylor-Hood element is inf-sup stable for all $k \geq 1$.
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A New Finite Element

We seek after an element pair that

- is inf-sup stable.
- gives better mass conservation.
- is not much more “expensive” than the Taylor-Hood pair.

Idea

Augment the pressure ansatz space Q_h^k of the Taylor-Hood element by local constant functions

$$Q_h^k := \{q \in L_0^2(\Omega) : q = q_k + q_0, q_k \in C(\bar{\Omega}), \\ q_k|_K \in \mathcal{Q}_k(K), q_0|_K \in \mathcal{Q}_0(K) \quad \forall K \in \mathcal{T}_h\}.$$

Remark

- The ansatz space for the velocities remains unchanged.
- The pressure becomes discontinuous.

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Conservation of Mass

Testing with an elementwise constant function yields

$$(\operatorname{div} u_h, q_h)_{L^2(\Omega)} \quad \forall q_h \in Q_h \Rightarrow \int_K \operatorname{div} u_h \, dx = 0$$

and the element pair is locally divergence free.

Nevertheless, the solution is in general not pointwise solenoidal:

$$\operatorname{div} \mathbf{V_h} \not\subset Q_h$$

Stability

Theorem (A. 2013)

Let $\mathcal{T}_h$ be a regular decomposition of Ω. Then the pair

$$\begin{aligned}
\mathbf{V_h} &= \{\mathbf{v} \in H_0^1(\Omega)^d : v|_K \in \mathcal{Q}_{k+1}(K)^d \quad \forall K \in \mathcal{T}_h\} \\
Q_h &= \{q \in L_0^2(\Omega) : q = q_k + q_0, q_k \in C(\bar{\Omega}), \\
&\qquad q_k|_K \in \mathcal{Q}_k(K), q_0|_K \in \mathcal{Q}_0(K) \quad \forall K \in \mathcal{T}_h\}
\end{aligned}$$

for $d = 2, 3$, $k \geq 1$ is inf-sup stable.

Remark

For the case with polynomial $\mathcal{P}_k$ see [BCGG12].

Macroelements

Definition (Macroelement spaces)

$$\mathbf{V_{0,M}} = \{\mathbf{v} \in H_0^1(M)^d : \mathbf{v} = \mathbf{w}|_M, \mathbf{w} \in \mathbf{V_h}\}$$
$$Q_M = \{q|_M, q \in Q_h\}$$

Definition (Spurious pressure modes)

$$N_M = \left\{ q \in Q_M : \int_M q \operatorname{div} \mathbf{v} \, dx = 0 \quad \forall \mathbf{v} \in \mathbf{V_{0,M}} \right\}$$

Macroelements

Theorem (Inf-sup condition for macroelements, [BBF08])

Let $\mathcal{M}_h$ be a macroelement decomposition of $\mathcal{T}_h$ such that

(H1) N_M *are the constant functions for all* $M \in \mathcal{M}_h$;

(H2) *each* $M \in \mathcal{M}_h$ *is part of an equivalence class;*

(H3) *there are finitely many equivalence classes, the number does not depend on* h;

(H4) *each element is part of finitely many macroelements, the number does not depend on* h;

(H5) *the inf-sup conditions between* V_h *and the space of elementwise constant functions holds.*

Then the spaces V_h *and* Q_h *satisfy the inf-sup condition.*

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Then the spaces V_h *and* Q_h *satisfy the inf-sup condition.*

Proof of the inf-sup stability.

Choose the macroelement partition by grouping together for each internal vertex the touching elements.

- (H2) and (H4) hold due to the choice of macroelements.
- (H3) is a consequence of the regularity assumption.
- $\mathcal{Q}_{k+1}/\mathcal{P}_k$ is inf-sup stable $\Rightarrow$ (H5).

In order to prove (H1) show:

- $p \in N_M \Rightarrow \nabla p|K = 0 \qquad \forall K \in M$
- $\nabla p = 0 \in M$

$\square$

Convergence

Theorem ([GR86])

If the spaces $\mathbf{V_h}$ and Q_h satisfy the inf-sup condition, the discrete Stokes problem is well-defined and for the approximation error holds

$$\|\mathbf{u}-\mathbf{u_h}\|_{\mathbf{V}} + \|p-p_h\|_Q \ \leq C \inf_{\substack{\mathbf{v_h}\in\mathbf{V_h} \\ q_h\in Q_h}} \left(\|\mathbf{u}-\mathbf{v_h}\|_{\mathbf{V}} + \|p-q_h\|_Q\right)$$

where the constant C is independent of h.

Convergence

Theorem ([GR86])

If the continuous solution satisfies the regularity assumption

$$\mathbf{u} \in [H^{k+2}(\Omega) \cap H_0^1(\Omega)]^d, \quad p \in H^{k+1}(\Omega) \cap L_0^2(\Omega),$$

then the convergence result

$$\|\mathbf{u} - \mathbf{u_h}\|_{\mathbf{V}} + \|p - p_h\|_Q \leq C_1 h^{k+1} \left(\|\mathbf{u}\|_{[H^{k+2}(\Omega)]^d} + \|p\|_{H^{k+1}(\Omega)} \right)$$

holds for the discrete solution $(\mathbf{u_h}, p_h)$ *of the discrete Stokes problem. If* Ω *is convex, we get*

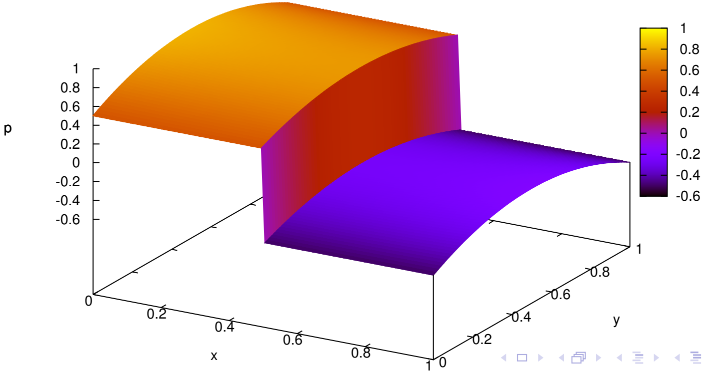
$$\|\mathbf{u} - \mathbf{u_h}\|_{[L^2(\Omega)]^d} \leq C_2 h^{k+2} \left(\|\mathbf{u}\|_{[H^{k+2}(\Omega)]^d} + \|p\|_{H^{k+1}(\Omega)} \right).$$

Reference Solution

As reference solution with discontinuous pressure we are using

$$\mathbf{u} = (\partial_y \psi_z, -\partial_x \psi_z), \qquad \psi_z = x^2(x-1)^2 y^2 (y-1)^2,$$

$$p = \begin{cases} y(1-y)exp(x-1/2)^2 + 1/2 & x \leq 1/2, \\ y(1-y)exp(x-1/2)^2 - 1/2 & x > 1/2 \end{cases}.$$



Reference Solution

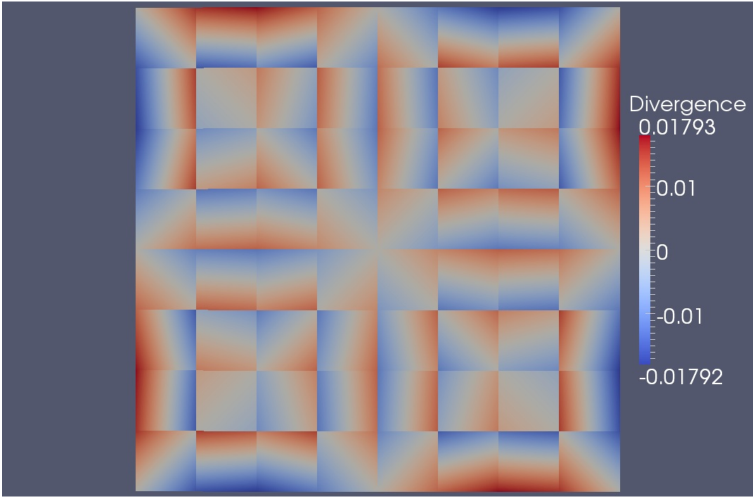
We choose the right hand side f such that the reference solution solves the Stokes problem.

$$\mathbf{f} = -\nu \Delta \mathbf{u} + \nabla p.$$

In order to take the discontinuity of the pressure into account we use the consistently modified right hand side

$$\langle \mathbf{f}, \mathbf{v} \rangle = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx + \sum_{K \in \mathcal{T}_h} \frac{1}{2} \int_{\partial K} [\![p]\!] \cdot \mathbf{v} \, dx.$$

Mass Conservation for the $\mathcal{Q}_2/(\mathcal{Q}_1 + \mathcal{Q}_0)$ Pair



Convergence

Table: Convergence results for the $\mathcal{Q}_2/(\mathcal{Q}_1 + \mathcal{Q}_0)$ element

$\lvert\mathcal{T}_h\rvert$	$\lVert\mathbf{u}-\mathbf{u_h}\rVert_{L^2}$		$\lvert\mathbf{u}-\mathbf{u_h}\rvert_{H^1}$		$\lVert p-p_h\rVert_{L^2}$	
4	$1.267 \cdot 10^{-3}$	-	$1.783 \cdot 10^{-2}$	-	$2.085 \cdot 10^{-2}$	-
16	$1.714 \cdot 10^{-4}$	2.89	$4.500 \cdot 10^{-3}$	1.99	$5.231 \cdot 10^{-3}$	2.00
64	$2.151 \cdot 10^{-5}$	2.99	$1.118 \cdot 10^{-3}$	2.01	$1.304 \cdot 10^{-3}$	2.00
256	$2.687 \cdot 10^{-6}$	3.00	$2.787 \cdot 10^{-4}$	2.00	$3.258 \cdot 10^{-4}$	2.00
1024	$3.357 \cdot 10^{-7}$	3.00	$6.962 \cdot 10^{-5}$	2.00	$8.146 \cdot 10^{-5}$	2.00
4096	$4.204 \cdot 10^{-8}$	3.00	$1.740 \cdot 10^{-5}$	2.00	$2.037 \cdot 10^{-5}$	2.00
16384	$6.026 \cdot 10^{-9}$	2.80	$4.352 \cdot 10^{-6}$	2.00	$5.101 \cdot 10^{-6}$	2.00

Table: Convergence results for the $\mathcal{Q}_2/\mathcal{Q}_1$ element

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
16384	$2.928 \cdot 10^{-5}$	1.50	$1.678 \cdot 10^{-2}$	0.50	$3.358 \cdot 10^{-2}$	0.50

Convergence

Table: Convergence results for the $\mathcal{Q}_3/(\mathcal{Q}_2 + \mathcal{Q}_0)$ element

$\lvert\mathcal{T}_h\rvert$	$\lVert\mathbf{u}-\mathbf{u_h}\rVert_{L^2}$		$\lvert\mathbf{u}-\mathbf{u_h}\rvert_{H^1}$		$\lVert p-p_h\rVert_{L^2}$	
4	$1.002 \cdot 10^{-4}$	-	$1.968 \cdot 10^{-3}$	-	$3.744 \cdot 10^{-4}$	-
16	$6.154 \cdot 10^{-6}$	4.03	$2.351 \cdot 10^{-4}$	3.07	$3.963 \cdot 10^{-5}$	3.24
64	$3.815 \cdot 10^{-7}$	4.01	$2.898 \cdot 10^{-5}$	3.02	$5.187 \cdot 10^{-6}$	2.93
256	$2.378 \cdot 10^{-8}$	4.00	$3.609 \cdot 10^{-6}$	3.01	$6.610 \cdot 10^{-7}$	2.97
1024	$1.486 \cdot 10^{-9}$	4.00	$4.506 \cdot 10^{-7}$	3.00	$8.372 \cdot 10^{-8}$	2.98

Instationary, Incompressible Navier-Stokes Equations

Unfiltered Navier-Stokes Equations

Find velocity $\mathbf{u}$ and pressure p in a polyhedral domain $\Omega \subset \mathbb{R}^d$:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \times (0, T), \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega \times (0, T), \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega \times (0, T), \\ \mathbf{u}(\cdot, 0) &= \mathbf{u_0} && \text{in } \Omega. \end{aligned}$$

$\Rightarrow$ additional nonlinearity and time derivative

Instationary, incompressible Navier-Stokes Equations

LES Navier-Stokes equations

$$
\begin{aligned}
\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}} + \nabla \bar{p} = \bar{\mathbf{f}} + \operatorname{div}(2(\nu + \nu_e)\mathbf{S}(\mathbf{u})) &\quad \text{in } \Omega \times (0, T), \\
\operatorname{div} \bar{\mathbf{u}} = 0 &\quad \text{in } \Omega \times (0, T), \\
\bar{\mathbf{u}} = \mathbf{0} &\quad \text{on } \partial\Omega \times (0, T), \\
\bar{\mathbf{u}}(\cdot, 0) = \bar{\mathbf{u_0}} &\quad \text{in } \Omega.
\end{aligned}
$$

with the symmetric strain-rate tensor

$$
S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_j}\right)
$$

and the turbulence model ν_e.

Implementation

The assembled equations can be written in a matrix form as

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}.$$

A : Diffusion, Advection, Reaction

B^T : Gradient B : Divergence

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} P^{-1} \begin{pmatrix} \widetilde{u} \\ \widetilde{p} \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

Use (F)GMRES with precondition matrix $\tilde{P}$ that approximates

$$P = \begin{pmatrix} A & B^T \\ 0 & S \end{pmatrix}$$

where $S = -BA^{-1}B^T$ is the Schur complement.

Approximation to S^{-1}

In cases where reaction is dominant S^{-1} can be approximated by

$$S^{-1} = (-BA^{-1}B^T)^{-1} \approx -\Delta^{-1}$$

- Poisson problem with homogeneous Neumann boundary conditions
- discontinuous pressure ansatz space

$\Longrightarrow$ Symmetric Interior Penalty Galerkin Method

SIPG - Symmetric Interior Penalty Galerkin Method

For the Poisson problem with homogeneous Neumann boundary conditions we get

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_\mathcal{I}} \alpha \, [\![u]\!] \cdot [\![v]\!] - \{\!\{\nabla u\}\!\} \cdot [\![v]\!] - [\![u]\!] \cdot \{\!\{\nabla v\}\!\} \, dx$$
$$= \int_\Omega f \cdot v \, dx.$$

Jump operator

$$[\![q]\!] = q^+ \mathbf{n}^+ + q^- \mathbf{n}^-$$

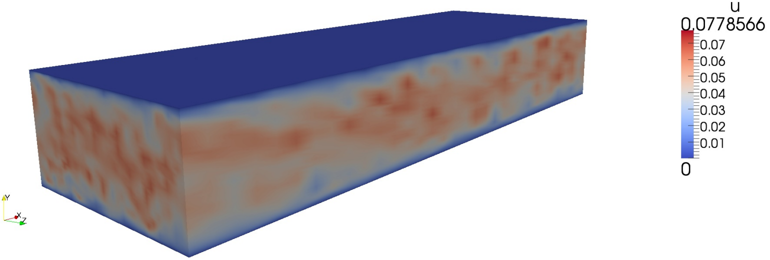
averaging operator

$$\{\!\{q\}\!\} = \frac{1}{2}(q^+ + q^-)$$

Turbulent Channel Flow

Turbulent Channel Flow

- Flow between two infinitely extended plates (x stream line, y anisotropic height, z width)
- $\Omega = 2\pi \times 2 \times \frac{4}{3}\pi$
- Random distortion of initial velocity (Reichelt's Law)
- $f = (0,0,0)$, $Re_\tau = 180$, $\nu_e(\mathbf{v}) \sim \frac{|\operatorname{tr}(\mathbf{S}^3(\mathbf{v}))|}{\operatorname{tr}(\mathbf{S}^2(\mathbf{v}))}$



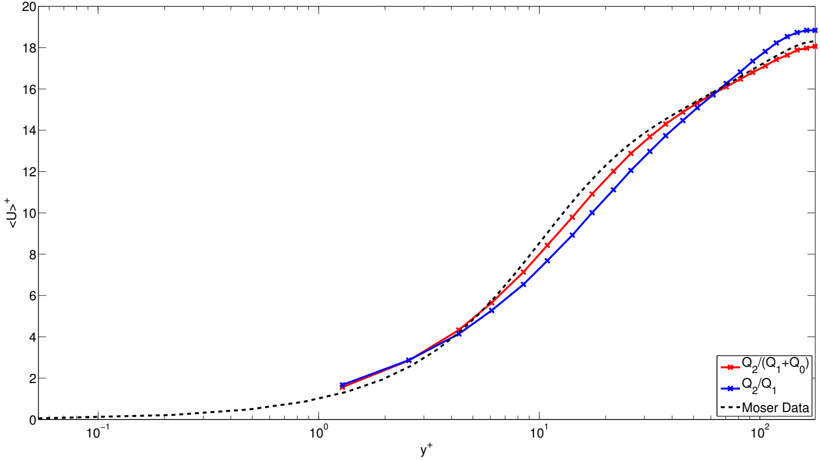
Characteristic Values

Characteristic Values

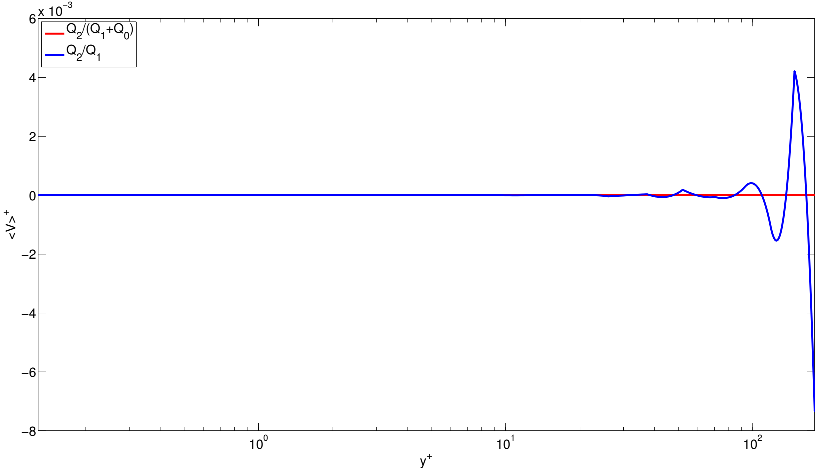
- mean value $\langle u \rangle$ averaged in time and space
- Reynolds decomposition $u = \langle u \rangle + u'$
- viscous length $y^+ = \nu \frac{\partial \langle u \rangle}{\partial y}|_{y=0} y$
- $u_\tau = \sqrt{\nu \frac{\partial \langle u \rangle}{\partial y}|_{y=0}}$
- $\langle u \rangle^+ = \frac{\langle u \rangle}{u_\tau}$

Reference data from [MKM99]

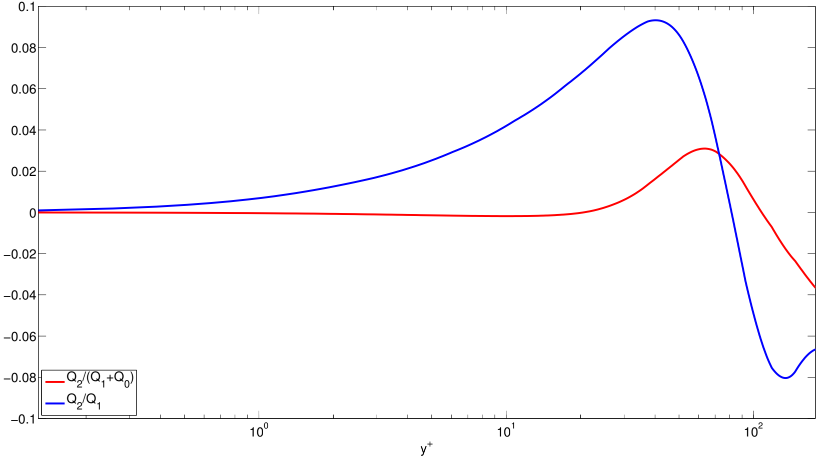
$\langle u \rangle^+$



$\langle v \rangle^+$



$\langle w \rangle^+$



Summary

Results

- The Q-DG0 element is inf-sup stable for tensor product polynomials,
- The same convergence results as for the Taylor-Hood element,
- Improved approximation to the mean profile of a turbulent channel flow by local mass conservation.

Challenge

- Choice of the preconditioner in the inner solver of the Navier-Stokes problem

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Thank you for your attention!