Notes on Applied Mathematics

Integration and Sobolev Spaces

Guido Kanschat

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Preface

This text is part of a set of notes I prepared for students and researchers. They mainly serve the purpose to be short and concise introductions to mathematical topics. They are provided as is and in the hope that they are useful. Nevertheless, I am always thankful for possible corrections and suggestions for enhancements.

The material in these notes is not my original research. Most of it is adapted from textbooks and research publications. While I am striving to give credit to the original authors wherever possible, I will be delighted to include more citations, also in order to improve the value of these notes as a reference.

Finally, if you find these notes useful for your own research and decide to cite results from them, I would be most flattered if you decided to cite them as

> Guido Kanschat. Notes on Applied Mathematics. Universität Heidelberg. 2013.

Note: yellow boxes indicate text which is missing in the current version and will be added soon.

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Chapter 1

Integration and Lebesgue spaces

1.1 The Lebesgue Integral

Introduction 1.1. This section introduces the Lebesgue integral mostly following the presentation in the monograph by Riesz and Sz.-Nagy [RSN90]. We deviate from this work only in two respects: first, we restrict the presentation to the results pertaining to the definition and properties of L^p-spaces. Second, we elaborate more on higher-dimensional integrals and modify the development of the one-dimensional case in order to be closer to higher dimensions.

1.1.1 Sets of measure zero

Introduction 1.2. Sets of measure zero constitute one of the most important concepts in integration and measure theory. Odd enough, it is possible to define them in an elementary way, which does not require any advanced measure theory.

Definition 1.3. A subset $Z \subset \mathbb{R}$ is called a **set of measure zero**, if for any $\varepsilon > 0$ there exist a finite or countable set of intervals I_k such that

$$Z \subset \bigcup_k I_k \quad \text{and} \quad \sum_k |I_k| < \varepsilon.$$

We also say that the set Z can be **covered** by a finite or countable union of intervals with total length less than ε.

Definition 1.4. Sets of meazure zero in higher dimensions are defined in a similar way, replacing the set of intervals I_k by cubes Q_k such that their total volume is less than ε.

Example 1.5. A set consisting of a single number $x \in \mathbb{R}$ is of measure zero, since for any $\varepsilon > 0$:

$$\{x\} \subset \left[x - \tfrac{\varepsilon}{2}, x + \tfrac{\varepsilon}{2}\right].$$

Lemma 1.6. *The finite or countable union of sets of measure zero is of measure zero.*

Proof. Let $\{Z_j\}_{j=1,2,\dots}$ be a finite or countable sequence of sets of measure zero. For each Z_j, let $\{I_{jk}\}$ be a set of intervals covering Z_j and having total length less than $2^{-k}\varepsilon$. Such a set exists according to the definition of a set of measure zero. Then,

$$Z = \bigcup_j Z_j \subset \bigcup_{jk} I_{jk} \quad \text{and} \quad \sum_{jk} |I_{jk}| < \varepsilon.$$

It remains to note that the index set jk is at most countable. □

Corollary 1.7. *The set $\mathbb{Q} \subset \mathbb{R}$ of rational numbers is of measure zero, since it is a countable union of points.*

Note 1.8. The preceding corollary implies that a dense subset of an interval in $\mathbb{R}$ can be covered by a system of intervals without covering the whole interval. Definitely, a remarkable statement.

Definition 1.9. A property is said to hold **almost everywhere** on a set M, if it holds on M, or at least on a set $M \setminus Z$, where Z is of measure zero.

1.1.2 Step functions and their integrals

Definition 1.10. We introduce **lattices** $\mathbb{Q}_n$[1] of $\mathbb{R}^d$ consisting of half open cubes

$$Q^{(n)}_{i_1,\dots,i_d} = \left]\tfrac{i_1}{2^n}, \tfrac{i_1+1}{2^n}\right] \times \left]\tfrac{i_2}{2^n}, \tfrac{i_2+1}{2^n}\right] \times \dots \times \left]\tfrac{i_d}{2^n}, \tfrac{i_d+1}{2^n}\right], \qquad i_k \in \mathbb{Z}.$$

The lattice is said to be of width 2^{-n}.

Note 1.11. Independent on the dimension d, the number of cubes in $\mathbb{Q}_n$ is countable. Thus, we can replace the multiple indices indicating the position of a cube in the lattice by a single enumeration index k. Furthermore, it is easy to see that the number of cubes in $\mathbb{Q}_n$ contained in a bounded set is finite, albeit depending on n.

Definition 1.12. A function f is called a **step function** on the lattice $\mathbb{Q}_n$, if f is constant on each cube Q_k, and if f is different from zero only on a finite number of cubes. We refer to the value of f on Q_k as $f(Q_k)$ or short f_k.

We denote by $\mathcal{S}$ the space of step functions.

[1] The letter $\mathbb{Q}$ with index always refers to a lattice and never to the rational numbers

Note 1.13. A step function f on a lattice $\mathbb{Q}_n$ is also a step function on any lattice $\mathbb{Q}_m$ with $m > n$, by simply choosing it to be the same constant on all the cubes of $\mathbb{Q}_m$ which are subsets of the same cube of $\mathbb{Q}_n$.

Therefore, further on, we can always compare two step functions by comparing them on the finer lattice used for their definition.

Definition 1.14. The **integral** of a step function f on $\mathbb{Q}_n$ is defined in the obvious way as

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \sum_{Q_k} f(q_k)|Q_k|,$$

where $|Q_k|$ denotes the volume of the cube Q_k. Since the sum in this definition is finite, the integral is finite.

Lemma 1.15 (Properties of the integral). *The integral of step functions is a linear operator, that is, for two step functions f and g and numbers $a, b \in \mathbb{R}$ holds*

$$\int_{\mathbb{R}^d} \big(af(x) + bg(x)\big)\,\mathrm{d}x = a\int_{\mathbb{R}^d} f(x) + b\int_{\mathbb{R}^d} g(x).$$

Furthermore, the integral is monotonic, that is, if for all $x \in \mathbb{R}^d$ holds $f(x) \le g(x)$, then holds

$$\int_{\mathbb{R}^d} f(x) \le \int_{\mathbb{R}^d} g(x).$$

Proof. Both properties follow from the fact that they hold for the values $f(Q_k)$ and $g(Q_k)$ and the summation operator. □

Definition 1.16. The **support** of a function is the set

$$\operatorname{supp} f = \overline{\left\{x \in \mathbb{R}^d \middle| f(x) \neq 0\right\}}. \tag{1.1}$$

A function f is said to have **finite support** or synonymously **compact support** if $\operatorname{supp} f$ is a bounded set.

Note 1.17. Since the support of a step function f consists of finitely many cubes, its support is finite.

The following two lemmas establish the close connection between the convergence of step functions almost everywhere and convergence of their integrals.

Lemma 1.18. *Let $\{\varphi_n\}_{n=1,\dots}$ be a monotonically decreasing sequence of nonnegative step functions on lattices $\mathbb{Q}_n$ converging to zero almost everywhere. Then,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi_n(x)\,\mathrm{d}x = 0.$$

Proof. First, we note that the assumptions imply that for all $n > 1$ holds

$$S := \operatorname{supp} \varphi_1 \supset \operatorname{supp} \varphi_n.$$

and thus the volume of the support of φ_n is bounded by that of S; let the volume of S be V.

Let now $\varepsilon > 0$ be arbitrarily small. Let Z be the set of measure zero, where the sequence does either not converge to zero, or where any of the functions φ_n is discontinuous. Let $\mathcal{J}_\varepsilon$ be an at most countable covering of this set of total volume less than ε according to Definitions 1.3 and 1.4.

Let J be the union of all elements in $\mathcal{J}_\varepsilon$. We note that, since the sequence is decreasing, for any $x \in S$ holds $\varphi_n(x) \le \varphi_1(x) \le M$ and thus

$$\int_J \varphi_n(x) \le \varepsilon M.$$

For any point $x \in \mathbb{R}^d \setminus J$ holds $\varphi_n(x) \to 0$ as $n \to \infty$. In particular, for n sufficiently large, $\varphi_n(x) \le \varepsilon$. Since φ_n is a step function, this holds not only for x, but for the whole cube containing x. By varying $x \in \mathbb{R}^d \setminus J$, we obtain an infinite set of such cubes, which we call $\mathcal{U}_\varepsilon$.

By their definition, the sets in the union of $\mathcal{J}_\varepsilon$ and $\mathcal{U}_\varepsilon$ cover the set S. And since S is compact, the Heine-Borel theorem says, that we can choose a finite subset from both of these systems, say $\breve{\mathcal{U}}_\varepsilon \cup \breve{\mathcal{J}}_\varepsilon$ to cover S. Let U be the union of all cubes in $\breve{\mathcal{U}}_\varepsilon$. Then, there is an index n_0, such that $\varphi_{n_0}(x) \le \varepsilon$ for all $x \in U$. Thus,

$$\int_U \varphi_n(x) < \varepsilon V, \qquad \forall n \ge n_0.$$

We conclude the proof by noting that for $n \ge n_0$

$$\int_S \varphi_n(x)\,\mathrm{d}x \le \int_U \varphi_n(x)\,\mathrm{d}x + \int_J \varphi_n(x)\,\mathrm{d}x < \varepsilon(M + V),$$

which can be made arbitrarily small by choosing ε small. □

Lemma 1.19. *Let $\{\varphi_n\}_{n=1,\dots}$ be a monotonically increasing sequence of step functions on lattices $\mathbb{Q}_n$ such that their integrals are uniformly bounded by a constant C:*

$$\int_{\mathbb{R}^d} \varphi_n(x)\,\mathrm{d}x \le C. \tag{1.2}$$

Then, the functions φ_n converge to a finite limit function φ almost everywhere in $\mathbb{R}^d$.

Proof. First, we observe that it is sufficient to consider sequences of nonnegative functions and thus positive constants C: otherwise, we consider the lemma for the sequence consisting of the functions $\varphi_n - \varphi_1$.

Let E_ε be the set of points x, where $\varphi_n(x) > C/\varepsilon$ for some n, and E_0 the set of points x, where $\varphi_n(x) \to \infty$. Obviously, $E_0 \subset E_\varepsilon$ for any $\varepsilon > 0$.

The set E_ε by its definition is an at most countable sequence of the cubes on which the step functions are defined. Therefore, the integral of φ_n over E_ε is defined and there holds

$$\frac{C}{\varepsilon} \sum_{Q_k \subset E_\varepsilon} |Q_k| \le \int_{E\varepsilon} \varphi_n(x)\,\mathrm{d}x \le \int_{\mathbb{R}^d} \varphi_n(x)\,\mathrm{d}x \le C.$$

From this, we deduce that the total volume of the cubes in E_ε does not exceed epsilon. Since this set of cubes covers E_0, we conclude that E_0 is of measure zero. □

Remark 1.20. Due to the monotonicity of the integral, the sequence on the left hand side of inequality (1.2) is monotonically increasing. Therefore, the integrals are actually converging to a finite value.

1.1.3 Definition of the integral

Introduction 1.21. In the previous section, we defined the integral of step functions and investigated some limits. The goal of this section is the extension of the integral to a wider class of functions. A first extension is almost immediately suggested by Lemma 1.19 and Remark 1.20. Namely, assigning as value of the integral of a function, which is the limit of an increasing sequence of step functions almost everywhere, the limit of their integrals. Nevertheless, it remains to prove that this yields a well defined integral, in particular, that its definition is independent of the actual choice of the sequence of step functions.

Lemma 1.22. *Let $\{\varphi_n\}$ be a monotonically increasing sequence of step functions with uniformly bounded integrals, converging to a function f almost everywhere. Let the same hold for a second sequence $\{\psi_n\}$ and the function g. Furthermore, let*

$$f(x) \le g(x) \quad \text{almost everywhere in } \mathbb{R}^d.$$

Then,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi_n(x)\,\mathrm{d}x \le \lim_{n\to\infty} \int_{\mathbb{R}^d} \psi_n(x)\,\mathrm{d}x. \tag{1.3}$$

Proof. First, we introduce the abbreviations

$$M_1 = \lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi_n(x)\,\mathrm{d}x \quad \text{and} \quad M_2 = \lim_{n\to\infty} \int_{\mathbb{R}^d} \psi_n(x)\,\mathrm{d}x.$$

Let φ_m be an arbitrary function in the first sequence. Then, by the assumptions, $h_n^+ = \varphi_m - \psi_n$, the positive part of the difference, converges to zero almost everywhere and decreases monotonically. Thus, by Lemma 1.18, its integrals tend to zero as well. We conclude

$$\int_{\mathbb{R}^d} \varphi_n(x)\,\mathrm{d}x - M_2 \le 0 \quad \text{or} \quad \int_{\mathbb{R}^d} \varphi_n(x)\,\mathrm{d}x \le M_2.$$

If now we let $n \to \infty$, we obtain $M_1 \le M_2$. □

Note 1.23. For two functions f and g as in the preceding Lemma with $f(x) = g(x)$ almost everywhere, we can repeat the argument of the proof and interchange the sequences. Thus, $M_1 = M_2$. This means in particular, that the limit process in this lemma uniquely defines the integral of the limit functions. Furthermore, it means that we can always modify a function on a set of measure zero without affecting its integral.

Definition 1.24. Let $\mathcal{S}^+$ be the set of functions which equal the limit of a sequence of monotonically increasing step functions with uniformly bounded integrals almost everywhere. For any function $f \in \mathcal{S}^+$, we define the integral as

$$\int_{\mathbb{R}^d} f(x)\,dx := \lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi_n(x)\,\mathrm{d}x,$$

where $\{\varphi_n\}$ is any monotonically increasing sequence of step functions converging to f almost everywhere.

Example 1.25. The function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is in $\mathcal{S}^+$ and its integral is zero. This is due to the fact that it is equal to the zero function almost everywhere.

Lemma 1.26. *Let h be a function which is the difference of two functions in $\mathcal{S}^+$, namely $h(x) = f_1(x) - f_2(x)$ almost everywhere. If for two other functions in $\mathcal{S}^+$ holds $h(x) = g_1(x) - g_2(x)$ almost everywhere, then*

$$\int_{\mathbb{R}^d} f_1(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} f_2(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} g_1(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} g_2(x)\,\mathrm{d}x, \tag{1.4}$$

that is, the difference of the integral does not depend on the actual choice of the two functions in the difference.

Definition 1.27. Let $\mathcal{L}$ be the class of functions[2] which can be written as a difference of two functions in $\mathcal{S}^+$ almost everywhere. For any function $h \in \mathcal{L}$, the **integral** is defined as

$$\int_{\mathbb{R}^d} h(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} g(x)\,\mathrm{d}x,$$

where f and g are any two functions such that $h(x) = f(x) - g(x)$ almost everywhere.

The class $\mathcal{L}$ is called the set of **integrable functions**.

1.1.4 Structure of the class of integrable functions

Introduction 1.28. The next question we will have to address is, which functions are integrable, and whether we are allowed to interchange limits of sequences of functions and their integrals. In particular, we will see that suitably bounded sequences of integrable functions have integrable limits.

Lemma 1.29. *For every integrable function f there exists a sequence of step functions $\{\varphi_n(x)\}$ converging to f almost everywhere, such that*

$$\int_{\mathbb{R}^d} \bigl|f(x) - \varphi_n(x)\bigr|\,\mathrm{d}x \to 0 \quad \textit{as} \quad n \to \infty.$$

Proof. This lemma is an immediate consequence of the definition of the class $\mathcal{L}$ as differences of functions in $\mathcal{S}^+$. In fact, let $f(x) = h(x) - g(x)$ with both functions in $\mathcal{S}^+$ and let $\{\psi_n(x)\}$ and $\{\varrho_n(x)\}$ be increasing sequences of step functions converging to $h(x)$ and $g(x)$ almost everywhere, respectively. By definition,

$$\int_{\mathbb{R}^d} \bigl|f(x) - \varphi_n(x)\bigr|\,\mathrm{d}x \le \int_{\mathbb{R}^d} \bigl(h(x) - \psi_n(x)\bigr)\,\mathrm{d}x + \int_{\mathbb{R}^d} \bigl(g(x) - \varrho_n(x)\bigr)\,\mathrm{d}x \to 0.$$

□

Theorem 1.30. *The class $\mathcal{L}$ of integrable functions is a vector space and the integral is linear, namely for functions $f, g \in \mathcal{L}$ and numbers $\alpha, \beta \in \mathbb{R}$ holds*

$$\int_{\mathbb{R}^d} \bigl(\alpha f(x) + \beta g(x)\bigr)\,\mathrm{d}x = \alpha \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x + \beta \int_{\mathbb{R}^d} g(x)\,\mathrm{d}x. \tag{1.5}$$

For functions $g(x)$ and $h(x)$ in $\mathcal{S}+$ are $\sup(g(x), h(x)$ and $\inf(g(x), h(x)$ in $\mathcal{S}+$. Furthermore, for a function $f \in \mathcal{L}$, its positive part f^+, its negative part f^-, and its absolute value $|f|$ are in $\mathcal{L}$.

[2] Here, the letter $\mathcal{L}$ is used in reference to the Lebesgue integral. We note that Lebesgue used the term "summable" instead of "integrable".

Proof. The linerarity of the integral follows immediately by taking corresponding linear combinations of step function sequences. Thus, $\mathcal{L}$ is a vector space. The same argument holds for the infimum and supremum of two functions in $\mathcal{S}^+$.

For the second part, we write $f(x) = h(x) - g(x)$ with both functions in $\mathcal{S}^+$, and note

$$\begin{aligned}
|f| &= \sup(g,h) - \inf(g,h) \\
f^+ &= \sup(g,h) - g \qquad &&= h - \inf(g,h) \\
f^- &= \sup(g,h) - h \qquad &&= g - \inf(g,h).
\end{aligned}$$

□

Definition 1.31. The **characteristic function** of a subset $\Omega \subset \mathbb{R}^d$ is χ_Ω. It is defined as

$$\chi_\Omega(x) = \begin{cases} 1 & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

The function f is said to be **integrable** on the set Ω, if $f\chi_\Omega$ is integrable (on $\mathbb{R}^d$). The **integral** over Ω of a function f is defined as

$$\int_\Omega f(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\chi_\Omega(x)\,\mathrm{d}x.$$

The **measure** of a bounded domain Ω is the integral of χ_Ω.

Corollary 1.32. *The integral is additive, namely for two disjoint and not necessarily bounded domains $\Omega_1 \subset \mathbb{R}^d$ and $\Omega_2 \subset \mathbb{R}^d$ and a function f such that $f\chi_{\Omega_1}$ and $f\chi_{\Omega_2}$ are integrable holds*

$$\int_{\Omega_1\cup\Omega_2} f(x)\,\mathrm{d}x = \int_{\Omega_1} f(x)\,\mathrm{d}x + \int_{\Omega_2} f(x)\,\mathrm{d}x.$$

Proof. This follows immediately from the linearity of the integral by observing $f(x)\chi_{\Omega_1\cup\Omega_2}(x) = f(x)\chi_{\Omega_1}(x) + f(x)\chi_{\Omega_2}(x)$. □

Remark 1.33. The space $\mathcal{S}^+$ and thus the space of integrable functions have been obtained by a completion process. We started out from the space of all test functions and then added to this space all monotonically increasing limits with respect to the topology "convergent almost everywhere". An important characteristic of this process is, that it is idempotent, that is, its repeated application does not change the result. We first verify this statement for $\mathcal{S}^+$. It is extended to integrable functions by the Beppo-Levi Theorem 1.35 below.

Lemma 1.34. *Let f_n be an increasing sequence of functions in $\mathcal{S}^+$. Assume further that the integrals of f_n are uniformly bounded by a number M. Then, the sequence f_n converges almost everywhere to a function $f \in \mathcal{S}^+$ and there holds*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n\,\mathrm{d}x = \int_{\mathbb{R}^d} f\,\mathrm{d}x.$$

Proof. For each n, let $\{\varphi_{nk}\}_{k=1,,2,\dots}$ be a sequence of step functions converging monotonically to f_n almost everywhere. We generate a type of diagonal sequence in $\{\varphi_{nk}\}$ by assigning

$$\psi_n(x) = \sup_{i\leq n} \varphi_{in}(x).$$

Since each of the sequences $\{\varphi_{nk}\}$ is increasing with respect to k, ψ_n must be increasing. Since they furthermore increase towards f_n, we have almost everywhere $\psi_n(x) \leq f_n(x)$. Thus,

$$\int_{\mathbb{R}^d} \psi_n \,\mathrm{d}x \leq \int_{\mathbb{R}^d} f_n \,\mathrm{d}x \leq M.$$

By Lemma 1.19, the sequence ψ_n converges towards a limit function f almost everywhere. On the other hand, for $n \leq k$,

$$\psi_n \geq \varphi_{nk}.$$

Therefore, by allowing $k \to \infty$, we obtain that

$$f_n(x) \leq f(x) \quad \text{almost everywhere.}$$

Hence, we have $\psi_n \leq f_n \leq f$ and $\psi_n \nearrow f$ almost everywhere, thus the limit of f_n must be equal to the one of ψ_n, namely f_n. The same argument holds for the integrals, and since

$$\int_{\mathbb{R}^d} \psi_n \,\mathrm{d}x \leq \int_{\mathbb{R}^d} f_n \,\mathrm{d}x \leq \int_{\mathbb{R}^d} f \,\mathrm{d}x,$$

and since the first integral converges to the last, so does the one in the center. □

Theorem 1.35 (Beppo-Levi). *Every monotonically increasing sequence $\{f_n(x)\}$ of integrable functions whose integrals have a common bound, converges almost everywhere to an integrable function $f(x)$, and the order of taking the limit and integrating can be reversed, that is*

$$\int_{\mathbb{R}^d} f(x) \,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x) \,\mathrm{d}x \tag{1.6}$$

Proof. Let us first rewrite the problem as

$$f_n(x) = f_0(x) + \sum_{k=1}^{n} g_k(x),$$

where $g_k(x) = f_k(x) - f_{k-1}(x)$. By the assumptions, the elements g_k of the series are nonnegative and integrable. Furthermore, since the integrals of f_n have a common bound, say M, the sequence of integrals is bounded by

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}^d} g_n(x) \,\mathrm{d}x \leq M + \int_{\mathbb{R}^d} |f_0(x)| \,\mathrm{d}x,$$

and thus convergent.

...

□

Theorem 1.36 (Lebesgue)**.** *If the sequence of integrable functions* $\{f_n(x)\}$ *converges to a function* $f(x)$ *almost everywhere, and if there exists an integrable function such that for all* n

$$\left|f_n(x)\right| \le g(x), \tag{1.7}$$

holds almost everywhere, then the function $f(x)$ *is integrable and*

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\,\mathrm{d}x.$$

Proof. ...

□

Example 1.37. Condition (1.7) in Lebesgue's Theorem is necessary, as can be seen from the following example. Let for $x \in \mathbb{R}$

$$f_n(x) = \begin{cases} (n+1)|x|^n & x \in [-1,1] \\ 0 & \text{else.} \end{cases}$$

The sequence converges to zero almost everywhere, but the integrals are two. Indeed, by changing the factor in front of x^n, the limit of the integrals can be made any value including ∞. On the other hand, if the factors are uniformly bounded, the limit is zero, as the theorem predicts.

The assumptions of Lebesgue's Theorem are too strong for some applications. Thus, the following lemma proves boundedness of the integral of the limit under weaker assumptions.

Lemma 1.38 (Fatou)**.** *If the functions* f_n *are nonnegative, integrable, and converge almost everywhere in* Ω *to a limit function* f*, and if furthermore the sequence of integrals*

$$\int_\Omega f_n(x)\,\mathrm{d}x,$$

is bounded, then f *is integrable and*

$$\int_\Omega f(x)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega f_n(x)\,\mathrm{d}x.$$

Proof. □

1.2 The Lebesgue function spaces L^p

1.2.1 Integral inequalities

Lemma 1.39 (Young's inequality). *For two arbitrary numbers a and b, and a positive number γ holds*

$$2|ab| \le \gamma a^2 + \frac{1}{\gamma} b^2. \tag{1.8}$$

Proof. From the binomial formulas we have

$$0 \le \begin{cases} (a+b)^2 & = a^2 + b^2 + 2ab, \\ (a-b)^2 & = a^2 + b^2 - 2ab, \end{cases}$$

and thus by bringing $2ab$ to the left,

$$\pm 2ab \le a^2 + b^2. \tag{1.9}$$

For $\gamma \neq 1$, the inequality is obtained by replacing a with γa and b with b/γ in (1.9) and observing that this does not change the left hand side. □

Lemma 1.40 (Cauchy-Bunyakovsky-Schwarz inequality[3]). *If functions f and g and their squares are integrable on a subset Ω of $\mathbb{R}^d$, so is their pointwise product fg, and there holds*

$$\int_\Omega f(x)g(x)\,\mathrm{d}x \le \sqrt{\int_\Omega f^2(x)\,\mathrm{d}x}\sqrt{\int_\Omega g^2(x)\,\mathrm{d}x}. \tag{1.10}$$

Proof. First, by Young's inequality we obtain for arbitrary γ that

$$2\int_\Omega f(x)g(x)\,\mathrm{d}x \le \gamma \int_\Omega f^2(x)\,\mathrm{d}x + \frac{1}{\gamma}\int_\Omega g^2(x)\,\mathrm{d}x.$$

Choosing γ such that both terms on the right are equal, namely

$$\gamma = \sqrt{\int_\Omega g^2(x)\,\mathrm{d}x} \Bigg/ \sqrt{\int_\Omega f^2(x)\,\mathrm{d}x},$$

the inequality is obtained. □

[3] Also known as Cauchy-Schwarz or Schwarz's inequality

Note 1.41. The Cauchy-Bunyakovsky-Schwarz inequality is an immediate extension of the Cauchy inequality for discrete sums

$$\sum_k a_k b_k \leq \sqrt{\sum_k a_k^2}\sqrt{\sum_k b_k^2} \tag{1.11}$$

Lemma 1.42 (Hölder's inequality)**.** *Assume that the functions $|f|^p$ and $|g|^q$ are integrable with $1 < p, q < \infty$ and*

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{1.12}$$

then the function fg is integrable on Ω and

$$\int_\Omega |fg| \,\mathrm{d}x \leq \sqrt[p]{\int_\Omega |f|^p \,\mathrm{d}x}\sqrt[q]{\int_\Omega |g|^q \,\mathrm{d}x}. \tag{1.13}$$

Lemma 1.43 (Minkowski's inequality)**.** *Let the functions $|f|^p$ and $|g|^p$ be integrable on Ω for $1 \leq p < \infty$. Then, the function $|f+g|^p$ is integrable on Ω and*

$$\sqrt[p]{\int_\Omega |f+g|^p \,\mathrm{d}x} \leq \sqrt[p]{\int_\Omega |f|^p \,\mathrm{d}x} + \sqrt[p]{\int_\Omega |g|^p \,\mathrm{d}x} \tag{1.14}$$

1.2.2 The real Hilbert spaces $L^2(\Omega)$

Introduction 1.44. For a bounded or unbounded subset $\Omega \subseteq \mathbb{R}^d$, the set of functions with squares, which are integrable with finite integrals forms a vector space. This is assured by the Minkowski's inequality (1.14). Our goal in this section is equipping this space with an inner product and a norm. Once a norm has been defined, we will show that the space obtained is complete, thus a Hilbert space. We start this section by the attempt to introduce an inner product.

Lemma 1.45. *Let $\Omega \subseteq \mathbb{R}^d$ be bounded or unbounded. The form*

$$\langle f, g \rangle = \int_\Omega f(x)g(x) \,\mathrm{d}x \tag{1.15}$$

is defined and bounded for functions f and g with integrable squares. It is bilinear, positive semidefinite, and symmetric.

Proof. According to the Cauchy-Bunyakovsky-Schwarz inequality (1.10), boundedness of $\langle f, g \rangle$ follows from boundedness of the integrals of f^2 and g^2. Linearity in f and g are an immediate consequence of the linearity of the integral in equation (1.5). Symmetry follows from the fact that we can change the order of f and g in the product under the integral. Finally, we have

$$0 \leq \int_\Omega f^2(x) \,\mathrm{d}x = \langle f, f \rangle .$$

□

Introduction 1.46. We have obtained $\langle f, f\rangle \ge 0$, but the definiteness of an inner product requires that $\langle f, f\rangle = 0$ implies $f = 0$. On the other hand, $\langle f, f\rangle = 0$ holds for any function, which is zero almost everywhere in Ω, but can have arbistrary values on a set of measure zero. Thus, we have no hope to define a definite inner product by integration on a function space. The solution to this dilemma is a modification of the function space by a little trick. Essentially, we turn around the definitions and define the relation $f = 0$ through the condition $\langle f, f\rangle = 0$. We will now do this in a mathematically sound way.

Lemma 1.47. *Let F be a set of functions on $\mathbb{R}^d$. We introduce the relation $f \simeq g$ if f and g differ at most on a set of measure zero, or*

$$f \simeq g \quad \Leftrightarrow f(x) - g(x) = 0 a.e. \tag{1.16}$$

This relation is an equivalence relation.

Proof. We have to show reflexivity, symmetry, and transitivity of "$\simeq$". Obviously, since $f(x) = f(x)$ for all x, we have $f \simeq f$. Similarly obvious from the definition is that $f \simeq g$ implies $g \simeq f$. Finally, let $f \simeq g$ and $g \simeq h$. Then,

$$f(x) - h(x) = \bigl(f(x) - g(x)\bigr) + \bigl(g(x) - h(x)\bigr).$$

Accordingly, the set on which f and h differ is at most the union of two set of measure zero, thus $f \simeq h$. □

Lemma 1.48. *The set of functions*

$$Z = \bigl\{f \big| f \equiv 0\bigr\},$$

where "0" is the function which is zero everywhere, is a vector space. Thus, for any vector space V of functions, the quotient set V/Z is a vector space.

Proof. The first part of the proof is obvious. Thus, for any element of V vector addition and scalar multiplication can be split into their components in Z and the remainder, which makes them well defined on the quotient set. □

Definition 1.49. Let V be the space of functions with bounded square integrals on a set $\Omega \subseteq \mathbb{R}^d$. Let Z be the equivalence class of the zero function according to the preceding lemmas. We define the vector space $L^2(\Omega)$ as the quotient space V/Z. The space $L^2(\Omega)$ is equipped with the inner product $\langle ., .\rangle$ according to equation (1.15) and the norm

$$\|f\| = \|f\|_2 := \sqrt{\langle f, f\rangle}. \tag{1.17}$$

Note 1.50. We say for two functions f and g in $L^2(\Omega)$ that "$f = g$" if the functions coincide almost everywhere. Thus, the missing definiteness of the inner product in Lemma 1.45 is obviously cured by considering equivalence classes. Thus, the norm is actually a norm, and $L^2(\Omega)$ for the moment is a pre-Hilbert space, that is, an inner product space, which is not necessarily complete.

Remark 1.51. Consistent with other authors, we usually refer to an element of $L^2(\Omega)$ as a **square integrable function** or L^2-function. Nevertheless, it is important to keep in mind that these elements are not functions at all. In particular, they do not assign a value $f(x)$ to a given point $x \in \mathbb{R}^d$, since $\{x\}$ is a set of measure zero, and thus we would be allowed to change this value any time. This observation is important whenever we deal with function spaces $L^2(\Omega)$ and similar objects. Later we will have to answer the question what kind of evaluation of an L^2-function is actually permitted.

Definition 1.52. We say that a sequence f_n in $L^2(\Omega)$ converges to an element $f \in L^2(\Omega)$ if $\|f_n - f\| \to 0$ as $n \to \infty$. A sequence f_n in $L^2(\Omega)$ is called a **Cauchy sequence**, if the **Cauchy criterion**

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \quad \forall m, n > n_\varepsilon : \|f_m - f_n\| < \varepsilon. \tag{1.18}$$

For the latter, we also say $\|f_m - f_n\| \to 0$ as $n, m \to \infty$.

Theorem 1.53 (Fischer-Riesz). *Let $\Omega \subseteq \mathbb{R}^d$. Then, $L^2(\Omega)$ is a Hilbert space, that is, it is complete, that is, every Cauchy sequence f_n in $L^2(\Omega)$ converges to an element $f \in L^2(\Omega)$.*

Proof. First, we note that the Cauchy criterion is necessary for convergence, since by the triangle inequality, we have for $m, n \to \infty$:

$$\|f_m - f_n\| \le \|f_m - f\| + \|f - f_n\| \to 0.$$

Now we assume that the Cauchy criterion holds. Then, there exists a sequence $n_1, n_2, \dots$ such that $\|f_{n_{k+1}} - f_{n_k}\| < 2^{-k}$. Now we have to distinguish between bounded domains Ω and unbounded domains. First, for bounded domains, the Cauchy-Bunyakovsky-Schwarz inequality implies that

$$\int_\Omega 1 \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right| \mathrm{d}x \le \sqrt{m(\Omega)} \, \|f_{n_{k+1}} - f_{n_k}\| \le \sqrt{m(\Omega)} 2^{-k},$$

hence the series

$$\sum_{k=1}^\infty \int_\Omega \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right| \mathrm{d}x$$

converges to a finite value. The sequence of functions defined by

$$s_m(x) = \sum_{k=1}^m \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right|,$$

is monotonically increasing. Furthermore, we have seen above that their integrals are uniformly bounded. Thus, by the Beppo-Levi theorem, $s_m(x)$ converges almost everywhere to an integrable function. This on the other hand implies that for almost every $x \in \Omega$ the sequence $f_{n_k}(x)$ is a Cauchy sequence and thus converges to a limit value $f(x)$.

If the domain Ω is unbounded and its measure is not finite, the above argument can be applied to any finite subdomain. Covering Ω with a countable sequence of such subdomains, we can likewise conclude convergence to a limit function f for almost every $x \in \Omega$.

Now we prove that $f \in L^2(\Omega)$. We observe that

$$\|f_{n_k}\| \le \|f_{n_1}\| + \|f_{n_k} - f_{n_1}\| \le \|f_{n_1}\| + \frac{1}{2}.$$

Thus, $\|f_{n_k}\|^2$ is uniformly bounded and as $|f_{n_k}(x)|^2 \to |f(x)|^2$ almost everywhere, Fatou's lemma asserts that $\|f\|$ is finite and accordingly $f \in L^2(\Omega)$.

It remains to show that $\|f_n - f\| \to 0$.

Show that norms converge

□

1.2.3 The real Banach spaces $L^p(\Omega)$

Introduction 1.54. We saw that the Cauchy-Bunyakovsky-Schwarz inequality has a generalization for exponents different from 2 in Hölder's inequality, further that Minkowski's inequality holds for arbitrary p with $1 \le p < \infty$. This suggests to define a norm through

$$\|f\|_p = \sqrt[p]{\int_\Omega |f|^p \,\mathrm{d}x}, \tag{1.19}$$

and it is indeed an easy exercise to prove the norm properties analogue to the previous subsection.

Definition 1.55. A function $f : \Omega \to \mathbb{R}$ is called **essentially bounded** from above, if there is a number a such that the set

$$(f > a) := \left\{x \in \Omega \middle| f(x) > a\right\},$$

has measure zero. The number a is called an **essential upper bound**. For an essentially bounded function, we define the **essential supremum** $\operatorname{esssup} f$ as the infimum of all essential upper bounds:

$$\operatorname{esssup} f = \inf_{a \in \mathbb{R}} (f > a) \text{ has measure zero.}$$

Definition 1.56. For $1 \leq p < \infty$, the space $L^p(\Omega)$ is the space of all functions such that $|f|^p$ is integrable with its norm defined in (1.19). The space $L^\infty(\Omega)$ is the space of all essentially bounded functions on Ω, and its norm is

$$\|f\|_\infty = \text{esssup}_{x \in \Omega} f$$

Note 1.57. For $p \neq 2$, the L^p-norm is not defined by an inner product, thus $L^p(\Omega)$ cannot be a Hilbert space.

Theorem 1.58. *The spaces $L^p(\Omega)$ are complete, thus, they are Banach spaces.*

Chapter 2

Weak derivatives and Sobolev spaces

Introduction 2.1. There are two fundamentally different definitions of Sobolev spaces, which are usually referred to as the spaces H^k and $W^{k,p}$. The first group is obtained by completing a space of continuously differentiable functions with respect to a given norm. The second definition relies on the introduction of distributional derivatives and then restricts the set of functions with such derivatives to those bounded with respect to a given norm. Thus, we can say that H^k approximates the desired space from the inside, while $W^{k,p}$ bounds it from the outside. An important result of modern analysis was the conclusion that both classes are actually the same.

Details on Sobolev spaces, including most of the material below can be found in [AF03]. A very condensed introduction is also in[GT98, Chapter 7].

2.1 The Sobolev spaces $H^1(\Omega)$

Introduction 2.2. These spaces are defined by first defining a norm for them, then completing for instance the space $\mathcal{C}^\infty$ with respect to this norm. This will lead to some difficulties with the involved symbols, which we will resolve in Section 2.2. We note that the problem of definiteness of the norm is the same as in the definition of $L^2(\Omega)$, which is, why we again have to take equivalence classes.

Definition 2.3. For functions $f, g \in \mathcal{C}^\infty(\Omega)$, we define the inner products, the H^1-seminorm $|.|_1$ and the H^1-norm $\|.\|_1$ as

$$\langle f, g\rangle_1 = \int_\Omega \nabla f \cdot \nabla g \,\mathrm{d}x, \qquad |f|_1 = \|\nabla f\|_0\,, \tag{2.1}$$

$$\langle\!\langle f, g\rangle\!\rangle = \langle f, g\rangle_0 + \langle f, g\rangle_1\,, \qquad \|f\|_1 = \|f\|_0 + |f|_1. \tag{2.2}$$

Here, $\langle .,. \rangle_0$ and $\|.\|_0$ refer to the inner product and norm in $L^2(\Omega)$, respectively.

Definition 2.4. First we compute the completion of $\mathcal{C}^\infty(\Omega)$ with respect to the norm $\|f\|_1$, that is, the set of limits of all Cauchy sequences with respect to the H^1-norm consisting of elements in $\mathcal{C}^\infty(\Omega)$ with uniformly bounded H^1-norm. The H^1-norm of the limit function is defined as the limit of the norms of the sequence.

The **Sobolev space** $H^1(\Omega)$ is the set of equivalence classes in this completion, where we say $f \simeq g$ if $\|f - g\|_1 = 0$.

Note 2.5. The space $\mathcal{C}^\infty(\Omega)$ in this definition could have been replaced by $\mathcal{C}^1(\Omega)$ with no different effect.

Example 2.6. The completion process in the definition above indeed yields functions which were not in $\mathcal{C}^\infty$. For instance, it is possible to construct a smooth sequence of functions converging to the function $f(x) = |x|$ with respect to the H^1-norm on $[-1, 1]$.

Definition 2.7. The space $\mathcal{C}_0^\infty(\Omega)$ is the space fo functions with ininitely many continuous derivatives and compact support in Ω.

Definition 2.8. The Sobolev space $H_0^1(\Omega)$ is obtained by completing the space $\mathcal{C}_0^\infty(\Omega)$ with respect to the H^1-norm and taking equivalence classes such that $f \simeq g$ if $\|f - g\|_1 = 0$.

Note 2.9. It remains to be shown that the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ are actually different.

2.2 Weak derivatives and the Sobolev spaces $W^{1,2}$

Introduction 2.10. When we introduced the space $H^1(\Omega)$, we started with a subspace of continuous functions and extended this space by adding the limits of Cauchy sequences. The result is a complete space with respect to the norm $\|.\|_1$. On the other hand, it is not clear whether this is the biggest function space for which this norm is finite. Therefore, in this section, we take the opposite approach: we define a derivative in a very broad sense and then reduce to those derivatives bounded with respect to the norm $\|.\|_1$.

Definition 2.11. If for a given function u there exists a function w such that

$$\int_\Omega w\varphi \,\mathrm{d}x = -\int_\Omega u\partial_i\varphi \,\mathrm{d}x, \qquad \forall \varphi \in \mathcal{C}_0^\infty(\Omega), \tag{2.3}$$

then we define $\partial_i u := w$ as the **distributional derivative** (partial) of u with respect to x_i. Similarly through integration by parts, we define distributional directional derivatives, distributional gradients etc.

Note 2.12. The formula (2.3) is the usual integration by parts. Therefore, whenever $u \in \mathcal{C}^1$ in a neighborhood of x, the distributional derivative and the usual derivative coincide.

Example 2.13. Let $\Omega = \mathbb{R}$ and $u(x) = |x|$. Intuitively, it is clear that the distributional derivative, if it exists, must be the **Heaviside function**

$$w(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0. \end{cases} \tag{2.4}$$

The proof that this is actually the distributional derivative is left to the reader.

Example 2.14. For the derivative of the Heaviside function in (2.4), we first observe that it must be zero whenever $x \neq 0$, since the function is continuously differentiable there. Now, we take a test function $\varphi \in \mathcal{C}^\infty$ with support in the interval $(-\varepsilon, \varepsilon)$ for some positive ε. Let $w'(x)$ be the derivative of w. Then, by integration by parts

$$\int_{-\varepsilon}^{\varepsilon} w(x)\varphi'(x)\,\mathrm{d}x = -\int_{-\varepsilon}^{0} w(x)'\varphi(x)\,\mathrm{d}x - \int_{0}^{\varepsilon} w(x)'\varphi(x)\,\mathrm{d}x + 2\varphi(0) = 2\varphi(0),$$

since $w'(x) = 0$ under both integrals. Thus, $w'(x)$ is an object which is zero everywhere except at zero, but its integral against a test function φ is nonzero. This contradicts our notion, that integrable functions can be changed on a set of measure zero without changing the integral. Indeed, w' is not a function in the usual sense, and we write $w'(x) = 2\delta(x)$, where $\delta(x)$ is the **Dirac δ-distribution**, which is defined by the two conditions

$$\begin{aligned} \delta(x) &= 0, \qquad \forall x \neq 0 \\ \int_{\mathbb{R}} \delta(x)\varphi(x)\,\mathrm{d}x &= \varphi(0), \quad \forall \varphi \in \mathcal{C}^0(\mathbb{R}). \end{aligned}$$

We stress that δ is not an integrable function, or a function at all.

Example 2.15. Take for instance $\Omega = [0, 1]$. It is known that Lipschitz-continuous functions on $\mathbb{R}$ are absolutely continuous and thus continuously differentiable almost everywhere, with their derivatives bounded by the Lipschitz constant, say L. Thus, the function itself is bounded on $[0, 1]$, say by M. Therefore, such a function f is weakly differentiable and its norm is bounded by

$$\|f\|_1 \leq L + M.$$

Definition 2.16. For a function $u \in L^2(\Omega)$, we call a distributional derivative a **weak derivative**, if the derivative is in $L^2(\Omega)$ as well. For such a **weakly differentiable** function, the seminorm $|.|_1$ and thus the norm $\|.\|_1$ is defined, if the gradient is understood in the distributional sense.

The space of weakly differentiable functions defined in this manner is the Sobolev space $W^{1,2}(\Omega)$, where the superscript one stands for the order of derivatives and the two is the exponent in the norm.

Remark 2.17. In this section and the previous section, we have seen two different definition of Sobolev spaces. One of the important theorems of modern analysis due to Meyers and Serrin [MS64] states that these two definitions actually specify the same object.

2.3 Higher order derivatives and different exponents

Introduction 2.18. Here we generalize the definitions of Sobolev spaces in Sections 2.1 and 2.2, respectively, to higher order derivatives. Most of this section will consist of introducing ugly notation, which is, why we postponed this. Mathematically, like for continuous derivatives, the same concepts as above are applied to obtain the new spaces.

Definition 2.19. A d-dimensional **multi-index** is a tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ with values α_i being nonnegative integers. We define partial derivatives of a function $u \in C^\infty$ with respect to a multi-index as

$$\partial_\alpha u(x) = \frac{\partial^\alpha}{\partial x^\alpha} u(x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \dots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} u(x_1, x_2, \dots, x_d).$$

The order of such a derivative is

$$|\alpha| = \sum_{i=1}^{d} \alpha_i.$$

Definition 2.20. Let u be integrable on $\Omega \subseteq \mathbb{R}^d$. If a function w exists such that

$$\int_\Omega w\varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u, \partial_\alpha \varphi \,\mathrm{d}x, \qquad \forall \varphi \in \mathcal{C}_0^\infty(\Omega), \tag{2.5}$$

then we call $\partial_\alpha u = w$ a distributional derivative of order $|\alpha|$ of u. If this derivative is in $L^2(\Omega)$, we call it a weak derivative.

Definition 2.21. The space $W^{k,2}(\Omega)$ is the space of functions $u \in L^2(\Omega)$ such that all distributional derivatives of order $|\alpha| \le k$ are in $L^2(\Omega)$. The $W^{k,2}$-seminorm and -norm are defined by

$$|f|_k = \sqrt{\sum_{|\alpha|=k} \|\partial_\alpha u\|^2}, \qquad \|f\|_k = \sqrt{\sum_{|\alpha|\le k} \|\partial_\alpha u\|^2}.$$

Remark 2.22. If we give up the notion of an inner product, all definitions above extend to the case where we replace L^2 by a space based on a norm with different exponent $1 \le p \le \infty$. This leads to the spaces $W^{k,p}(\Omega)$.

2.4 Properties of Sobolev spaces

Introduction 2.23. For continuously differentiable functions, the inclusion $\mathcal{C}^{k+1} \subset \mathcal{C}^k$ is obvious. The same inclusion holds for $W^{k+1,p}$ and $W^{k,p}$ by definition. Continuous functions are obviously continuous on smooth submanifolds, but Sobolev functions are a priori not even defined there. Nevertheless, we will see below, that there are operators which allow us to define traces of Sobolev functions on lower dimensional submanifolds, and even embeddings into spaces of continuous functions.

Definition 2.24. Let $\Omega \subset \mathbb{R}^d$. For the Sobolev space $W^{k,p}(\Omega)$, we define the **Sobolev number**

$$\sigma = k - \frac{d}{p}. \tag{2.6}$$

For the following theorems, verification is required when the cases $p = 1$ and $p = \infty$ are included, when equality of the Sobolev numbers is sufficient and when not, and which assumptions on the domains are reasonable.

Theorem 2.25. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Given two Sobolev spaces $W^{k_1,p_1}(\Omega)$ and $W^{k_2,p_2}(\Omega)$ with $1 \le p_1, p_2 < \infty$. If $\sigma_1 \ge \sigma_2$, then there exists a continuous embedding $W^{k_1,p_1}(\Omega) \hookrightarrow W^{k_2,p_2}(\Omega)$ such that*

$$\|u\|_{W^{k_2,p_2}(\Omega)} \le C \|u\|_{W^{k_1,p_1}(\Omega)}.$$

Theorem 2.26. *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ be a bounded Lipschitz domain and $\Omega_2 \subset \overline{\Omega}_1$ a smooth submanifold of dimension d_2. Then, if $\sigma_1 \ge \sigma_2$, there exists a continuous trace operator from $W^{k_1,p_1}(\Omega_1) \hookrightarrow W^{k_2,p_2}(\Omega_2)$, such that*

$$\|u\|_{W^{k_2,p_2}(\Omega_2)} \le C \|u\|_{W^{k_1,p_1}(\Omega_1)}.$$

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