Notes on Applied Mathematics

Discontinuous Galerkin methods

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Chapter 1

Discontinuous Galerkin methods

1.1 Nitsche's method

Introduction 1.1. Let us consider the inhomogeneous Dirichlet boundary value problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= u^D && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

We already know that for $u^d \equiv 0$ we can discretize this problem by choosing a finite element space $V_h \subset H^1_0(\Omega)$. For general u^D, there are two options:

1. Compute an arbitrary "lifting" $u^D \in H^1(\Omega)$ such that it is equal to u^D on the boundary and compute $w = u - u^D \in H^1_0(\Omega)$ as the solution to the weak formulation
$$\int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x + \int_\Omega \nabla u^D \cdot \nabla v \,\mathrm{d}x. \tag{1.2}$$
2. Compute an interpolation or projection u^D_h of the boundary data u^D. Then, eliminate each row of the matrix corresponding to a degree of freedom on the boundary. In particular, let i be the index of such a degree of freedom and let k be an index corresponding to an interior degree of freedom not constraint by a boundary condition, but such that $a_{ik} \neq 0 \neq a_{ki}$. Then, replace the rows
$$\begin{aligned} \cdots + a_{ii}u_i + \cdots + a_{ik}u_k + \cdots &= f_i \\ \cdots + a_{ki}u_i + \cdots + a_{kk}u_k + \cdots &= f_k \end{aligned} \tag{1.3}$$
by the rows
$$\begin{aligned} u_i \qquad\qquad\qquad\qquad &= u^D_i \\ \cdots + 0 + \cdots + a_{kk}u_k + \cdots &= f_k - a_{ki}u^D_i \end{aligned} \tag{1.4}$$

The first option introduces the complication of finding a function in $H^1(\Omega)$, which cannot be achieved automatically. The second can be implemented in an automatic way, but it complicates code, in particular for nonlinear problems.

A completely different approach modifying the bilinear form was first suggested in the 60s and then perfected by Joachim Nitsche in 1971. In this section, we motivate this method and derive its error estimates. Its key feature is the transition from $V_h \subset H^1_0(\Omega)$ to $V_h \subset H^1(\Omega)$.

Introduction 1.2. If we simply derive our weak formulation in $H^1(\Omega)$, we end up with an additional boundary term

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = -\int_\Omega \Delta u v \, \mathrm{d}x + \int_{\partial\Omega} \partial_n u v \, \mathrm{d}s. \tag{1.5}$$

Thus, we obtain the natural boundary condition $\partial_n u = 0$, which is not consistent with the original BVP. The first step for deriving Nitsche's method is subtracting this boundary term on both sides. The result is the equation

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} \partial_n u v \, \mathrm{d}s = \int_\Omega f v \, \mathrm{d}x \qquad \forall v \in H^1(\Omega). \tag{1.6}$$

We observe that the left hand side vanishes for any constant function u. Thus, we do not have unique solvability and we will have to fix this problem. Furthermore, the boundary data u^D does not appear in this formulation. We enforce $u = u^D$ in our formulation by a so called "penalty term" with penalty parameter α, modifying (1.6) to

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} \partial_n u v \, \mathrm{d}s + \int_{\partial\Omega} \alpha u v \, \mathrm{d}s \\ = \int_\Omega f v \, \mathrm{d}x + \int_{\partial\Omega} \alpha u^D v \, \mathrm{d}s \qquad \forall v \in H^1(\Omega). \end{aligned} \tag{1.7}$$

Integrating by parts, we see that u is a solution to this weak formulation. Following Nitsche, we make one additional modification which restores the symmetry of our form. We obtain the weak formulation

$$a_h(u, v) = f_h(v) \qquad \forall v \in H^1(\Omega), \tag{1.8}$$

where

$$\begin{aligned} a_h(u, v) &= \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} \partial_n u v \, \mathrm{d}s - \int_{\partial\Omega} \partial_n v u \, \mathrm{d}s + \int_{\partial\Omega} \alpha u v \, \mathrm{d}s, \\ f_h(v) &= \int_\Omega f v \, \mathrm{d}x - \int_{\partial\Omega} \partial_n v u^D \, \mathrm{d}s + \int_{\partial\Omega} \alpha u^D v \, \mathrm{d}s. \end{aligned} \tag{1.9}$$

We abbreviate this equation to

Note 1.3. Unfortunately, the problem (1.9) is not well-posed for any finite parameter α. Thus, it cannot be used to determine $u \in H^1(\Omega)$. Nevertheless, we can establish well-posedness on discrete spaces V_h in order to compute a discrete solution u_h and use the fact that u is already determined by the continuous problem. Our immediate goals are thus:

1. Establish the assumptions of the Lax-Milgram theorem on V_h, which in this case involves a suitable new norm for measuring the error.

2. Establish a relation between the discrete and continuous solution replacing Galerkin orthogonality.

3. Deriving error estimates in suitable norms.

Notation 1.4. From now on, we will use the inner product notation

$$(u, v) \equiv \int_\Omega uv \, \mathrm{d}x \qquad (\nabla u, \nabla v) \equiv \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x,$$

on Ω as well as

$$\langle u, v \rangle \equiv \int_{\partial\Omega} uv \, \mathrm{d}s,$$

on its boundary.

Definition 1.5. A discrete problem (1.9) is called **consistent**, if for the solution u of the BVP (1.1) there holds

$$a_h(u, v_h) = f_h(v_h) \qquad \forall v_h \in V_h. \tag{1.10}$$

Corollary 1.6. *Let $u \in H^1(\Omega)$ be the weak solution to* (1.1) *in the sense of* (1.2)*. Then, the discrete problem* (1.9) *is consistent.*

Proof. Since $u \in H^1(\Omega)$ and $v_h \in V_h$, all boundary terms in $a_h(u, v - h)$ and $f_h(v_h)$ are well-defined and consistency follows from $u = u^D$ in the sense of $L^2(\partial\Omega)$. □

Definition 1.7. The problem dependent norm used for the analysis of Nitsche's method is defined by

$$|||v|||^2 = (\nabla v, \nabla v) + \langle \alpha v, v \rangle . \tag{1.11}$$

This choice is justified by

Lemma 1.8. *Let $V_h \subset V = H^1(\Omega)$ be a piecewise polynomial finite element space on a shape-regular mesh $\mathbb{T}_h$. Then, if α sufficiently large, there exist constants M and γ such that*

$$a_h(u_h, v_h) \le M \left|\!\left|\!\left| u_h \right|\!\right|\!\right| \left|\!\left|\!\left| v_h \right|\!\right|\!\right|$$
$$a_h(u_h, u_h) \ge \gamma \left|\!\left|\!\left| u_h \right|\!\right|\!\right|^2 .$$

Proof. Key for the proof is the inverse trace estimate

$$|v|_{H^1(\partial T)} \le ch^{-1/2} |v|_{H^1(T)},$$

which holds with a constant c depending on shape regularity and polynomial degree. Thus, for a cell T at the boundary, there holds

$$\begin{aligned} |\langle \partial_n u_h, v_h \rangle_{\partial T \cap \partial\Omega}| &\le ch^{-1/2} |u_h|_{H^1(T)} \left\| v_h \right\|_{L^2(\partial T \cap \partial\Omega)} \\ &\le \tfrac{1}{4} |u_h|^2_{H^1(T)} + \tfrac{c^2}{h_T} \left\| v_h \right\|_{L^2(\partial T \cap \partial\Omega)} , \end{aligned}$$

We apply this to the lower bound to obtain

$$a_h(u_h, u_h) \ge \left(1 - \tfrac{1}{2}\right) |u_h|^2_{H^1(\Omega)} + \left(\alpha - \tfrac{2c^2}{h_T}\right) \left\| u_h \right\|^2_{L^2(\partial\Omega)} .$$

Choosing

$$\alpha(x) = \frac{\alpha_0}{h(x)} \ge \frac{4c^2}{h(x)}, \tag{1.12}$$

where $h(x)$ is the size of the cell such that $x \in \partial T$, we obtain

$$a_h(u_h, u_h) \ge \frac{1}{2} \left|\!\left|\!\left| u_h \right|\!\right|\!\right|^2 .$$

The proof of the upper bound follows the same fashion. □

Corollary 1.9. *Let α be chosen according to equation* (1.12)*. Then, the discrete problem* (1.8) *has a unique solution $u_h \in V_h$.*

Proof. According to the previous lemma, the lemma of Lax-Milgram applies to the bilinear form $a_h(.,.)$. For the right hand side $f_h(.,)$, we have again because of trace estimates in $H^1(\Omega)$ and inverse estimates in V_h

$$f_h(v) \le c \left(\left\| f \right\|_{0,\Omega} + \left\| u^D \right\|_{H^1(\Omega)} \right) \left|\!\left|\!\left| v \right|\!\right|\!\right| .$$

□

Theorem 1.10. *Let $u \in H^{k+1}(\Omega)$ with $k \geq 1$ be the solution of (1.1) and let $u_h \in V_h$ be the solution to (1.8) and let the assumptions of Lemma 1.8 hold. Let furthermore $\{\mathbb{T}_h\}$ be a family of quasi-uniform, shape-regular meshes of maximal cell diameter h, and let the shape function spaces contain the polynomial space P_k. Then, there holds*

$$\|\!|u - u_h|\!\| \leq ch^k |u|_{k+1,\mathbb{T}_h}. \tag{1.13}$$

Proof. We begin with the triangle inequality

$$\|\!|u - u_h|\!\| \leq \|\!|u - I_h u|\!\| + \|\!|I_h u - u_h|\!\|.$$

The interpolation error can be estimated by

$$\begin{aligned} |u - I_h u|_{1,\Omega} &\leq h^k \, |u|_{2,\Omega}, \\ |u - I_h u|_{0,\partial\Omega} &\leq h^{3/2} \, |u|_{2,\Omega}. \\ |u - I_h u|_{1,\partial\Omega} &\leq h^{1/2} \, |u|_{2,\Omega}. \end{aligned} \tag{1.14}$$

The second and third estimate actually require some deeper arguments from functional analysis, which is beyond the scope of this class. It involves an intuitive notion of Sobolev spaces with non-integer derivatives. Allowing such spaces, the trace estimate becomes

$$\|u\|_{1/2,\partial\Omega} \leq c \, \|u\|_{1,\Omega}. \tag{1.15}$$

For the remaining error term, we use V_h-ellipticity of the discrete form and consistency to obtain

$$\gamma \, \|\!|I_h u - u_h|\!\|^2 \leq a_h(I_h u - u_h, I_h u - u_h) = a_h(I_h u - u, I_h u - u_h). \tag{1.16}$$

Using Young's inequality, we estimate the right hand side with $\varepsilon_h = u - I_h u$ and $\eta_h = I_h u - u_h$ on each boundary cell T with boundary edge E by

$$\begin{aligned} |(\nabla\varepsilon_h, \nabla\eta_h)_T| &\leq \tfrac{1}{\gamma} \, |\varepsilon_h|^2_{1,T} + \tfrac{\gamma}{4} \, |\eta_h|^2_{1,T}, \\ |\langle\alpha\varepsilon_h, \eta_h\rangle_E| &\leq \tfrac{2}{\gamma} \left\|\sqrt{\alpha}\varepsilon_h\right\|^2_{0,E} + \tfrac{\gamma}{8} \left\|\sqrt{\alpha}\eta_h\right\|^2_{0,E} \\ |\langle\varepsilon_h, \partial_n\eta_h\rangle_E| &\leq \tfrac{1}{\gamma} \left\|\sqrt{\alpha}\varepsilon_h\right\|^2_{0,E} + \tfrac{\gamma}{4} \, |\eta_h|^2_{1,T} + \tfrac{\gamma}{4} \left\|\sqrt{\tfrac{\alpha_0}{h_T}}\eta_h\right\|^2_{0,E} \\ |\langle\partial_n\varepsilon_h, \eta_h\rangle_E| &\leq \tfrac{2h_T}{\gamma\alpha_0} \, |\varepsilon_h|^2_{0,E} + \tfrac{\gamma\alpha_0}{8h_T} \left\|\eta_h\right\|^2_{0,E}. \end{aligned} \tag{1.17}$$

Adding these over all cells, we obtain

$$
\begin{aligned}
&\gamma \left|\!\left|\!\left| I_h u - u_h \right|\!\right|\!\right|^2 \\
&\le \tfrac{\gamma}{2} \left|\!\left|\!\left| I_h u - u_h \right|\!\right|\!\right|^2 \left(\tfrac{1}{\gamma} \left| u - I_h u \right|_{1,\Omega}^2 + \tfrac{3}{\gamma} \left\| \sqrt{\alpha}(u - I_h u) \right\|_{0,E}^2 + \tfrac{2h_T}{\gamma\alpha_0} \left| u - I_h u \right|_{0,E}^2 \right),
\end{aligned}
$$

and thus, by the interpolation estimate (1.14)

$$
\left|\!\left|\!\left| I_h u - u_h \right|\!\right|\!\right|^2 \le \tfrac{2}{\gamma} c h^{2k} \left| u \right|_{k+1,\Omega}^2 .
$$

□

Theorem 1.11. *Assume in addition to the assumptions of Theorem 1.10 that the adjoint problem*

$$
\begin{aligned}
-\Delta z &= u - u_h \quad \text{in } \Omega \\
z &= 0 \qquad\quad \text{on } \partial\Omega.
\end{aligned}
\tag{1.18}
$$

admits elliptic regularity, namely

$$
\left| z \right|_2 \le c \left\| u - u_h \right\|_0 . \tag{1.19}
$$

Then, the solutions u and u_h admit the estimate

$$
\left\| u - u_h \right\|_0 \le c h^{k+1} \left| u \right|_{k+1} . \tag{1.20}
$$

Proof. Due to symmetry of the discrete bilinear form, we have adjoint consistency:

$$
a_h(v_h, z) = (u - u_h, v_h) \qquad \forall v_h \in V_h. \tag{1.21}
$$

From here, we proceed like in the continuous case:

$$
\left\| u - u_h \right\|_0^2 = a_h(u - u_h, z) = a_h(u - u_h, z - I_h z).
$$

Using the same derivation as in the previous theorem, we obtain the result. □

Remark 1.12. We chose the norm defined in (1.11) for our energy norm analysis in Theorem 1.10. We could have done the same using the operator norm $\sqrt{a_h(v,v)}$. In fact, Lemma 1.8 states that both norms are equivalent. Therefore, we chose the one involving less terms and providing for simpler interpolation estimates.

The "triple norm" notation $|\!|\!|.|\!|\!|$ is very common as a notation for problem adjusted norms.